* Note:-

① $\int e^{x}\,dx = e^{x}$

② $\int e^{-x}\,dx = -e^{-x}$

③ $\int e^{ax}\,dx = \frac{e^{ax}}{a}$, where a is constant

④ $\int e^{-ax}\,dx = \frac{e^{-ax}}{-a}$, where a is constant

QueJ The kms x in thousands of kms which car owners get with a certain kind of tyre is a random variable having probability density function -

$$F(x) = \begin{cases} \frac{1}{20} e^{-x/20}, & \text{For } x > 0 \\ 0, & \text{For } x \leq 0 \end{cases}$$

Find the probabilities that one of these tyres will last -

① at most 10,000 kms

② anywhere from 16,000 to 24,000 kms

③ at least 30,000 kms.

Solution:- Let random variable x denote the kms (in thousands) with a certain kind of tyre.

① The probability that one of these tyres will last at most 10,000 kms is given by

$$\therefore P(X \le 10) = \int_0^{10} F(x)\,dx = \int_0^{10} \frac{1}{20} e^{-x/20}\,dx$$

$$= \frac{1}{20}\int_0^{10} e^{-x/20}\,dx = \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_0^{10}$$

$$= \frac{1}{20} \times -\frac{20}{1}\left[e^{-x/20}\right]_0^{10}$$

$$= -1\left[e^{\frac{-10}{20}} - e^{\frac{-0}{10}}\right] = -1\left[e^{-0.5} - e^{0}\right]$$

$$= -1\left[0.6065 - 1\right] = -0.6065 + 1$$

$$= 1 - 0.6065 = 0.3935$$

$$\therefore \boxed{P(X \le 10) = 0.3935}$$

(ii) The probabilities that one of these tyres will last anywhere from 16,000 to 24,000 kms is given by

~~P(16~~

$$P(16 \le X \le 24) = \int_{16}^{24} F(x)\,dx = \int_{16}^{24} \frac{1}{20} e^{-x/20}\,dx$$

$$\therefore P(16 \le X \le 24) = \frac{1}{20}\int_{16}^{24} e^{-x/20}\cdot dx$$

$$= \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_{16}^{24}$$

$$= \frac{1}{20} \times -\frac{20}{1}\left[e^{-x/20}\right]_{16}^{24}$$

$$= -1\left[e^{\frac{-24}{20}} - e^{\frac{-16}{20}}\right]$$

$$= -1\left[e^{-1.2} - e^{-0.8}\right]$$

$$= -1\left[0.3012 - 0.4493\right]$$

$$= -0.3012 + 0.4493 = 0.4493 - 0.3012$$

$$= 0.1481$$

$$\therefore \boxed{P(16 \le X \le 24) = 0.1481}$$

(iii) The probability that one of these tyres will last at least 30,000 kms is given by

$$P(X \geq 30) = \int_{30}^{\infty} F(x)\,dx$$

$$= \int_{30}^{\infty} \frac{1}{20} e^{-\frac{x}{20}}\,dx = \frac{1}{20}\int_{30}^{\infty} e^{-x/20}\,dx.$$

$$= \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_{30}^{\infty} = \frac{1}{20} \times \frac{-20}{1}\left[e^{-x/20}\right]_{30}^{\infty}$$

$$= -1\left[e^{-\infty/20} - e^{-30/20}\right]$$

$$= -1\left[e^{-\infty} - e^{-1.5}\right] = -1\left[0 - e^{-1.5}\right]$$

$$= e^{-1.5} = 0.2231$$

$$\therefore \boxed{P(X \geq 30) = 0.2231}$$

Que] A petrol pump is supplied with petrol once a day. If it's daily volume of sales (X) in thousands of litres is distributed by

$$f(x) = 5(1-x)^4, \quad 0 \le x \le 1$$

What must be the capacity of it's tank in order that the probability that it's supply will be exhausted in a given day shall be 0.01 ?

<u>Solution:-</u> Let the capacity of the tank (in thousand's litres) be "a" such that

$$P(X \geq a) = 0.01$$

$$\Rightarrow \int_a^1 f(x)\,dx = 0.01$$

$$\Rightarrow \int_a^1 5(1-x)^4\,dx = 0.01$$

$$\Rightarrow 5\int_a^1 (1-x)^4\,dx = 0.01$$

$$\Rightarrow 5\left[\frac{(1-x)^5}{-5}\right]_a^1 = 0.01$$

$$\Rightarrow \frac{5}{-5}\left[(1-x)^5\right]_a^1 = 0.01$$

$$\Rightarrow -1\left[(1-1)^5-(1-a)^5\right]=0.01$$

$$\Rightarrow -1\left[0^5-(1-a)^5\right]=0.01$$

$$\Rightarrow -1\left[0-(1-a)^5\right]=0.01$$

$$\Rightarrow (1-a)^5=0.001$$

$$\Rightarrow (1-a)^5=\frac{1}{100}$$

$$\Rightarrow 1-a=\left(\frac{1}{100}\right)^5$$

$$\Rightarrow 1-\left(\frac{1}{100}\right)^5=a$$

$$\Rightarrow 1-0.3981=a \Rightarrow 0.6019=a$$

$$\therefore \boxed{a=0.6019}$$

Hence, capacity of the tank

$= 0.6019 \times 1000$ litres

$= 601.9$ litres

* Two - Dimensional Distribution Function:-

The distribution function of the two-dimensional random variable (X,Y) is a real valued function F defined for all real x & y by the relation -

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

* Marginal Distribution Function:-

The marginal probability function of X and Y are given respectively as follows -

$$F_X(x) = \begin{cases} \sum_y P_{XY}(x,y) & \text{---- For discrete variable} \\ \int_{-\infty}^{\infty} F_{XY}(x,y)\,dy & \text{--- For continuous variable} \end{cases}$$

$$F_Y(y) = \begin{cases} \sum_x P_{XY}(x,y) & \text{---- For discrete variable} \\ \int_{-\infty}^{\infty} F_{XY}(x,y)\,dx & \text{----- For continuous variable} \end{cases}$$

The marginal density functions of X and Y can be obtained in the following manner also.

$$F_X(x) = \frac{dF_X(x)}{dx} = \int_{-\infty}^{\infty} F_{XY}(x,y)\,dy$$

$$F_Y(y) = \frac{dF_Y(y)}{dy} = \int_{-\infty}^{\infty} F_{XY}(x,y)\,dx$$

* Stochastic Independence :-

Two random variables x and y with joint probability density function $F_{xy}(x,y)$ and marginal probability density function $F_x(x)$ and $g_y(y)$ respectively are said to be stochastically independent if and only if -

$$F_{xy}(x,y) = F_x(x) g_y(y)$$

Que] The joint probability distribution of two random variables x and y is given by

$P(x=0, y=1) = \frac{1}{3}$, $P(x=1, y=-1) = \frac{1}{3}$ and
$P(x=1, y=1) = \frac{1}{3}$

Find
① marginal distribution of x and y
② The conditional probability distribution of x given y=1.

Solution :-

y	x			marginal
	-1	0	1	y
-1	0	0	1/3	1/3
0	0	0	0	0
1	0	1/3	1/3	2/3
marginal x	0	1/3	2/3	1

From above table,

marginal distribution of X is –

values of X (x)	-1	0	1
P(X=x)	0	1/3	2/3

marginal distribution of Y is –

values of Y (y)	-1	0	1
P(Y=y)	1/3	0	2/3

② The conditional probability distribution of X given Y=1 is

Y	X			marginal (Y)
	-1	0	1	
-1	0	0	1/3	1/3
0	0	0	0	0
1	0	1/3	1/3	2/3
marginal (X)	0	1/3	2/3	1

The conditional probability distribution of X given Y is

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$\therefore \; P(X=-1 \mid Y=1) = \frac{P(X=-1, Y=1)}{P(Y=1)} = \frac{0}{2/3} = 0$$

$$P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}$$

Thus, the conditional probability of X given $Y=1$ is

Values of $X = x$	-1	0	1
$P(X=x \mid Y=1)$	0	$1/2$	$1/2$

Example 5·33. *For the adjoining bivariate probability distribution of X and Y, find :*

(i) $P(X \le 1, Y = 2)$,

(ii) $P(X \le 1)$,

(iii) $P(Y \le 3)$, *and*

(iv) $P(X < 3, Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution. The marginal distributions are given below :

X \ Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

(*i*) $P(X \le 1, Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2) = 0 + \frac{1}{16} = \frac{1}{16}$

(*ii*) $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

(*iii*) $P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$.

(*iv*) $P(X < 3, Y \le 4) = P(X = 0, Y \le 4) + P(X = 1, Y \le 4) + P(X = 2, Y \le 4)$

$$= \left(\frac{1}{32} + \frac{2}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}\right) = \frac{9}{16}.$$

Example 5·34. *For the joint probability distribution of two random variables X and Y given below :*

X \ Y	1	2	3	4	Total
1	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{10}{36}$
2	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{9}{36}$
3	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{8}{36}$
4	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{9}{36}$
Total	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	1

Find (i) the marginal distributions of X and Y, and

(ii) conditional distribution of X given the value of Y = 1 and that of Y given the value of X = 2.

Solution. The marginal distribution of X is defined as :

$$P(X = x) = \sum_y P(X = x, Y = y)$$

$$\therefore \quad P(X=1) = \sum_y P(X=1, Y=y)$$
$$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$$
$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}.$$

Similarly $P(X=2) = \sum_y P(X=2, Y=y) = \frac{9}{36}$; $P(X=3) = \sum_y P(X=3, Y=y) = \frac{8}{36}$

and $P(X=4) = \sum_y P(X=4, Y=y) = \frac{9}{36}$.

Similarly, we can obtain the marginal distribution of Y.

MARGINAL DISTRIBUTION OF X

Values of X, x	1	2	3	4
$P(X=x)$	$\frac{10}{36}$	$\frac{9}{36}$	$\frac{8}{36}$	$\frac{9}{36}$

MARGINAL DISTRIBUTION OF Y

Values of Y, y	1	2	3	4
$P(Y=y)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$

(*ii*) The conditional probability function of X given Y is defined as follows :

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}. \text{ Therefore}$$

$$\therefore \quad P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{4/36}{11/36} = \frac{4}{11}$$

$$P(X=2 \mid Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}.$$

$$P(X=3 \mid Y=1) = \frac{P(X=3, Y=1)}{P(Y=1)} = \frac{5/36}{11/36} = \frac{5}{11}$$

$$P(X=4 \mid Y=1) = \frac{P(X=4, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

Hence the conditional distribution of X given $Y=1$ is :

x :	1	2	3	4
$P(X=x \mid Y=1)$:	$\frac{4}{11}$	$\frac{1}{11}$	$\frac{5}{11}$	$\frac{1}{11}$

Similarly, we can obtain the conditional distribution of Y for $X=2$ as given below :

y :	1	2	3	4
$P(Y=y \mid X=2)$:	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{9}$

Example 5·35. *A two-dimensional r.v. (X, Y) have a bivariate distribution given by :*

$$P(X=x, Y=y) = \frac{x^2+y}{32}, \text{ for } x = 0, 1, 2, 3 \text{ and } y = 0, 1.$$

Find the marginal distributions of X and Y.

(b) A two-dimensional r.v. (X, Y) have a joint probability mass function :

$$p(x, y) = \frac{1}{27}(2x+y), \text{ where } x \text{ and } y \text{ can assume only the integer values 0, 1 and 2.}$$

Find the conditional distribution of Y for X = x.

Solution. (*a*) We have

X → / Y ↓	0	1	2	3	Marginal distribution of Y, $P(Y=y)$
0	0	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{9}{32}$	$\frac{14}{32}$
1	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{18}{32}$
Marginal distribution of X, $P(X=x)$	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{9}{32}$	$\frac{19}{32}$	1

The marginal probability distribution of X is given by :

$$P(X=x)=\sum_y P(X=x, Y=y)$$ and is tabulated in last row of above table.

The marginal probability distribution of Y is given by :

$$P(Y=y)=\sum_x P(X=x, Y=y)$$ and is tabulated in last column of above table.

(*b*) The joint probability function :

$$p_{XY}(x,y)=\frac{1}{27}(2x+y)\ ;\ x=0,1,2;\ y=0,1,2$$

gives the following table of joint probability distribution of X and Y.

JOINT PROBABILITY DISTRIBUTION $p(x,y)$ OF X AND Y

Y → / X ↓	0	1	2	$f_X(x)$
0	0	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{3}{27}$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$\frac{9}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{15}{27}$

For example, $p(0,0)=\frac{1}{27}(0+2\times 0)=0,\quad p(1,0)=\frac{1}{27}(0+2\times 1)=\frac{2}{27}$;

$p(2,0)=\frac{1}{27}(0+2\times 2)=\frac{4}{27}$; and so on,

The conditional distribution of Y for $X=x$ is given by :

$$P_{Y|X}(Y=y \mid X=x)=\frac{p_{XY}(x,y)}{p_X(x)}$$ and is obtained in the adjoining table .

CONDITIONAL DISTRIBUTION OF Y FOR $X=x$

X → / Y ↓	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$
2	$\frac{4}{15}$	$\frac{5}{15}$	$\frac{6}{15}$

6·2. MATHEMATICAL EXPECTATION OR EXPECTED VALUE OF A RANDOM VARIABLE.

Once we have constructed the probability distribution for a random variable, we often want to compute the mean or expected value of the random variable. The *expected value* of a discrete random variable is a weighted average of all possible values of the random variable, where the weights are the probabilities associated with the corresponding values. The mathematical expression for computing the expected value of a discrete random variable X with probability mass function (*p.m.f.*) $f(x)$ is given below :

$$\boxed{E(X) = \sum_x x\, f(x), \text{ (for discrete r.v.)}} \quad \text{... (6·1)}$$

The mathematical expression for computing the expected value of a continuous random variable X with probability density function (*p.d.f.*) $f(x)$ is, however, as follows :

$$\boxed{E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx, \text{ (for continuous r.v.)}} \quad \text{... (6·1a)}$$

provided the right, hand integral in (6·1*a*) or series in (6·1) is absolutely convergent, *i.e.*, provided

$$\int_{-\infty}^{\infty} |\, x f(x)\, |\, dx = \int_{-\infty}^{\infty} |\, x\, |\, f(x)\, dx < \infty \quad \text{... (6·2)}$$

or

$$\sum_x |\, x f(x)\, | = \sum_x |\, x\, |\, f(x) < \infty \quad \text{... (6·2a)}$$

6·4. PROPERTIES OF EXPECTATION

Property 1. Addition Theorem of Expectation.

If X and Y are random variables, then $E(X+Y) = E(X) + E(Y)$, ... (6·10)
provided all the expectations exist.

Proof. Let X and Y be continuous *r.v.*'s with joint *p.d.f.* $f_{XY}(x, y)$ and marginal *p.d.f.*'s $f_X(x)$ and $f_Y(y)$ respectively. Then by def.,

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx \quad \text{...(6·11)} \quad \text{and} \quad E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy \quad \text{...(6·12)}$$

$$E(X+Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y) f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{XY}(x, y)\, dx\, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \right] dx + \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \right] dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx + \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$= E(X) + E(Y) \quad \text{[On using (6·11) and (6·12)]}$$

6·5. PROPERTIES OF VARIANCE

If X is a random variable, then $V(aX + b) = a^2 V(X)$*, where a and b are constants.*

Proof. Let $Y = aX + b$. Then $E(Y) = aE(X) + b$

$$\therefore \quad Y - E(Y) = a[X - E(X)]$$

Squaring and taking expectation of both sides, we get

$$E[Y - E(Y)]^2 = a^2 E[X - E(X)]^2$$

$\Rightarrow$ $V(Y) = a^2 V(X)$ or $V(aX + b) = a^2 V(X)$,

where $V(X)$ is written for variance of X.

Cor. (*i*) If $b = 0$, then $V(aX) = a^2 V(x)$ ... (6·29*a*)

$\Rightarrow$ Variance is not independent of change of scale.

(*ii*) If $a = 0$, then $V(b) = 0$ $\Rightarrow$ Variance of a constant is zero. ... (6·29*b*)

(*iii*) If $a = 1$, then $V(X + b) = V(X)$ ... (6·29*c*)

$\Rightarrow$ Variance is independent of change of origin.

6·6. COVARIANCE

If X and Y are two random variables, then covariance between them is defined as

$$Cov(X,Y) = E[\{X - E(X)\}\{Y - E(Y)\}] \quad ... (6·30)$$

$$= E[XY - X E(Y) - Y E(X) + E(X) E(Y)]$$

$$= E(XY) - E(Y) E(X) - E(X) E(Y) + E(X) E(Y)$$

$$= E(XY) - E(X) E(Y) \quad ... (6·30a)$$

If X and Y are independent then $E(XY) = E(X) E(Y)$ and hence in this case

$$Cov(X, Y) = E(X) E(Y) - E(X) E(Y) = 0 \quad ... (6·30b)$$

Remarks 1. $Cov(aX, bY) = E[\{aX - E(aX)\}\{bY - E(bY)\}]$

$$= E[a\{X - E(X)\} b\{Y - E(Y)\}]$$

$$= ab\, E[\{X - E(X)\}\{Y - E(Y)\}]$$

$$= ab \text{ Cov}(X, Y) \quad ... (6·31)$$

2. $\text{Cov}(X + a, Y + b) = \text{Cov}(X, Y)$ ... (6·31*a*)

3. $\text{Cov}\left(\dfrac{X - \bar{X}}{\sigma_X}, \dfrac{Y - \bar{Y}}{\sigma_Y}\right) = \dfrac{1}{\sigma_X \sigma_Y} \text{Cov}(X, Y)$ ... (6·31*b*)

4. Similarly, we shall get :

$$\text{Cov}(aX + b, cY + d) = ac\, Cov(X, Y) \quad ... (6·31c)$$

$$\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z) \quad ... (6·31d)$$

$$\text{Cov}(aX + bY, cX + dY) = ac\sigma_x^2 + bd\sigma_Y^2 + (ad + bc) \text{Cov}(X, Y) \quad ... (6·31e)$$

... $\text{Cov}(X, Y) = 0$ [c.f. (6·30b)]. However, the converse is not

Example 6·1. *Let X be a random variable with the following probability distribution :*

x :	-3	6	9
$P(X = x)$:	$1/6$	$1/2$	$1/3$

Find E (X) and E (X^2) and using the laws of expectation, evaluate E $(2X + 1)^2$.

Solution. $E(X) = \Sigma\, x\, p(x) = (-3) \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = \frac{11}{2}$

$$E(X^2) = \Sigma x^2\, p(x) = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = \frac{93}{2}$$

$$\therefore \quad E(2X+1)^2 = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1 = 4 \times \frac{93}{2} + 4 \times \frac{11}{2} + 1 = 209.$$

Example 6·3. *In four tosses of a coin, let X be the number of heads. Tabulate the 16 possible outcomes with the corresponding values of X.. By simple counting, derive the probability distribution of X and hence calculate the expected value of X.*

Solution. Let H represent a head, T a tail and X, the random variable denoting the number of heads.

S. No.	Outcomes	No. of Heads (X)	S. No.	Outcomes	No. of Heads (X)
1	HHHH	4	9	HTHT	2
2	HHHT	3	10	THTH	2
3	HHTH	3	11	THHT	2
4	HTHH	3	12	HTTT	1
5	THHH	3	13	THTT	1
6	HHTT	2	14	TTHT	1
7	HTTH	2	15	TTTH	1
8	TTHH	2	16	TTTT	0

The random variable X takes the values 0, 1, 2, 3 and 4. Since, from the above table, we find that the number of cases favourable to the coming of 0, 1, 2, 3 and 4 heads are 1, 4, 6, 4 and 1 respectively, we have

$$P(X=0)=\frac{1}{16},\ P(X=1)=\frac{4}{16}=\frac{1}{4},\ P(X=2)=\frac{6}{16}=\frac{3}{8},\ P(X=3)=\frac{4}{16}=\frac{1}{4},\ P(X=4)=\frac{1}{16}.$$

The probability distribution of X can be summarized as follows :

x :	0	1	2	3	4
$p(x)$:	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$$\therefore \quad E(X) = \sum_{x=0}^{4} x\,p(x) = 1\cdot\frac{1}{4} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{16} = \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4} = 2.$$

Example 6·4. *An urn contains 7 white and 3 red balls. Two balls are drawn together, at random from this urn. Compute the probability that neither of them is white. Find also the probability of getting one white and one red ball. Hence compute the expected number of white balls drawn.*

Solution. Let X denote the number of white balls drawn. The probability distribution of X is obtained as follows :

x :	0	1	2
$p(x)$:	$\frac{{}^3C_2}{{}^{10}C_2} = \frac{1}{15}$	$\frac{{}^7C_1 \times {}^3C_1}{{}^{10}C_2} = \frac{7}{15}$	$\frac{{}^7C_2}{{}^{10}C_2} = \frac{7}{15}$

Then expected number of white balls drawn is :

$$E(X) = 0 \times \frac{1}{15} + 1 \times \frac{7}{15} + 2 \times \frac{7}{15} = \frac{21}{15}.$$