# Module - 3

## Introduction to probability & conditional probability

* **Basic Terminologies related to probability :-**

① **Random Experiment :-** If in each trial of an experiment conducted under identical conditions, the outcome is not unique, but may be any one of the possible outcomes, then such an experiment is called a random experiment.

For e.g. tossing a coin, throwing a die, selecting a card from a pack of playing cards etc. In all these cases, there are a number of possible results which can occur but there is an uncertainty as to which one of them will actually occur.

② **Outcome :-** The result of a random experiment will be called an outcome.

③ **Trial and Event :-** Any particular performance of a random experiment is called a trial.

The outcome or combination of outcomes are termed as events.

For e.g. If a coin is tossed repeatedly, the result is not unique. We may get any of the two faces, head or tail. Thus tossing a coin is a random experiment or trial and getting of a head or tail is an event.

④ **Exhaustic Events or cases :-** The total number of possible outcomes of a random experiment is known as the exhaustic events or cases.

For e.g. In tossing of a coin, the exhaustic events are head and tail.

⑤ Favourable Events or cases:— The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event.

For e.g. In drawing a card from a pack of cards the number of cases favourable to drawing a red card is 26.

* Definition of probability:—

In a random experiment or a trial results in "n" exhaustive, mutually exclusive and equally likely outcomes (or cases), out of which "m" are favourable to the occurrence of an event E, then the probability "p" of occurrence (or happening) of E, usually denoted by P(E) is given by —

$$\boxed{p = P(E) = \frac{\text{Number of Favourable cases}}{\text{Total number of exhaustic cases}} = \frac{m}{n} = \frac{n(E)}{n(S)}}$$

Here $m \geq 0$, $n > 0$ and $m \leq n$

Also $\boxed{0 \leq P(E) \leq 1}$

The non-happening of the event E is called the complementary event of E and is denoted by $\bar{E}$ or $E^c$. and

$$\boxed{P(E) + P(\bar{E}) = 1}$$

$$\therefore \boxed{P(E) = 1 - P(\bar{E}) \quad \text{or} \quad P(\bar{E}) = 1 - P(E)}$$

QUE] Two unbiased dice are thrown. Find the probability that -

1. both the dice show the same number
2. the first dice shows 6.
3. the total of the numbers on the dice is 8.
4. the total of the numbers on the dice is greater than 8.
5. The total of the numbers on the dice is 13.

**Answer:-** In a random throw of two dice, since each of the six faces of one dice can be associated with each of six faces of the other dice, number of favourable cases is $6 \times 6 = 36$ and given as -

$\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

(1,1), (2,1), (3,1), (4,1), (5,1), (6,1)
(1,2), (2,2), (3,2), (4,2), (5,2), (6,2),
(1,3), (2,3), (3,3), (4,3), (5,3), (6,3),
(1,4), (2,4), (3,4), (4,4), (5,4), (6,4).
(1,5), (2,5), (3,5), (4,5), (5,5), (6,5)
(1,6), (2,6), (3,6), (4,6), (5,6), (6,6)

$\therefore$ Here

size of sample space $n(S) = 36$

1. $E_1$ = The favourable cases that both dice shows the same number.

$\therefore n(E_1) = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$\therefore \quad n(E_1) = 6$

$$\therefore P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(2) Let $E_2$ = First die shows 6.

$\therefore n(E_2) = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$\therefore n(E_2) = 6$

$\therefore P(E_2) = \frac{n(E_2)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

(3) Let $E_3$ = The total of the numbers on the dice is 8.

$\therefore E_3 = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$\therefore n(E_3) = 5$

$\therefore P(E_3) = \frac{n(E_3)}{n(S)} = \frac{5}{36}$

(4) Let $E_4$ = The total of the numbers on the dice is greater than 8.

$\therefore E_4 = \{(3,6), (6,3), (4,5), (5,4), (4,6), (6,4), (5,5),$
$(5,6), (6,5), (6,6)\}$

$\therefore n(E_4) = 10$

$\therefore P(E_4) = \frac{n(E_4)}{n(S)} = \frac{10}{36} = \frac{5}{18}$

(5) Let $E_5$ = The total of the numbers on the dice is 13.

$\therefore E_5 = \{\ \}$.

$\therefore n(E_5) = 0$

$\therefore P(E_5) = \frac{n(E_5)}{n(S)} = \frac{0}{36} = 0$.

Que 7 From 25 tickets, marked with first 25 numerals, one is drawn at random. Find the chance that -

(i) It is multiple of 5 or 7.

(ii) It is multiple of 3 or 7.

**Answer:-**

Total number of tickets = 25.

And they market with first 25 numbers

i.e. 1, 2, 3, 4, 5, - ------, 21, 22, 23, 24, 25.

$\therefore$ size of sample space $n(S) = 25$

(i) Let $E_1$ = An event that it is a multiple of 5 or 7.

$\therefore E_1 = \{5, 10, 15, 20, 25, 7, 14, 21\}$

$\therefore n(E_1) = 8$

$\therefore \boxed{P(E_1) = \frac{n(E_1)}{n(S)} = \frac{8}{25}}$

(ii) Let $E_2$ = An event that it is a multiple of 3 or 7.

$\therefore E_2 = \{3, 6, 9, 12, 15, 18, 21, 24, 7, 14\}$

$\therefore n(E_2) = 10$

$\therefore \boxed{P(E_2) = \frac{n(E_2)}{n(S)} = \frac{10}{25} = \frac{2}{5}}$

Que] In a random toss of three coins, Find the probability that -

1. Three heads occurs
2. Three tails occurs
3. Exactly one head occurs
4. Exactly two head occurs
5. At least one head occur
6. At least two heads occurs.

Answer:- In a random toss of three coins, the sample space S is given by

$$S = \{H, T\} \times \{H, T\} \times \{H, T\}$$

$$S = \{HH, HT, TH, TT\} \times \{H, T\}$$

$$S = \{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}$$

$$\therefore \boxed{n(S) = 8}$$

1. Let $E_1$ = An even that, three heads occurs.

$$\therefore E_1 = \{HHH\}$$

$$\therefore n(E_1) = 1$$

$$\therefore \boxed{P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1}{8}}$$

2. Let $E_2$ = An event that, three tails occurs.

$$\therefore E_2 = \{TTT\}$$

$$\therefore n(E_2) = 1$$

$$\therefore \boxed{P(E_2) = \frac{n(E_2)}{n(S)} = \frac{1}{8}}$$

(3) Let $E_3$ = An event that, exactly one head occur.

$\therefore E_3 = \{HTT, THT, TTH\}$

$\therefore n(E_3) = 3$

$$\therefore \boxed{P(E_3) = \frac{n(E_3)}{n(S)} = \frac{3}{8}}$$

(4) Let $E_4$ = An event that, exactly two heads occurs.

$\therefore E_4 = \{HHT, HTH, THH\}$

$\therefore n(E_4) = 3$

$$\therefore \boxed{P(E_4) = \frac{n(E_4)}{n(S)} = \frac{3}{8}}$$

(5) Let $E_5$ = An event that, Atleast one head occur.

$\therefore \cancel{E_5 = \{HHH, HHT, HTH, THH\}}$

$\therefore \cancel{n(E_5) = 4}$

$\therefore E_5 = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$

$\therefore n(E_5) = 7$

$$\therefore \boxed{P(E_5) = \frac{n(E_5)}{n(S)} = \frac{7}{8}}$$

(6) Let $E_6$ = An event that, atleast two heads occurs.

$\therefore E_6 = \{HHH, HHT, HTH, THH\}$

$\therefore n(E_6) = 4$

$$\therefore \boxed{P(E_6) = \frac{n(E_6)}{n(S)} = \frac{4}{8} = \frac{1}{2}}$$

Que] H.W.  In a random throw of two dice, Find the probability that –

1. sum of points on two dice is 5.
2. sum of points on two dice is 6.
3. sum of points on two dice is even
4. sum of points on two dice is odd
5. sum of points on two dice is greater than 12
6. sum of points on two dice is divisible by 3.

## * Acceptable Assignment of probabilites :-

Let $e_1, e_2, \ldots e_n$ be mutually disjoint and exhaustive outcomes of a random experiment, so that it's sample space is

$$S = \{e_1, e_2, \ldots e_n\}$$

To each elementary event $e_i, i = 1, 2, \ldots n$ belonging to $S$, let us assign a real number called the probability of the elementary event $e_i$ and denoted by $P(e_i)$ such that

(i) The probability of each elementary event is a non-negative real number.

$$\therefore \boxed{P(e_i) \geq 0} \quad \text{For } i = 1, 2, \ldots, n.$$

(ii) $$\sum_{i=1}^{n} P(e_i) = 1$$

## * Natural Assigment of probabilites :-

An assignment of probabilites is called natural (or equiprobable) assigment if each elementary event is assigned the same probability.

If $e_i$ and $e_j$ are two elementary events of the sample space S, then in the case of natural assignment of probabilities

$$\boxed{P(e_i) = P(e_j)} , \forall i \neq j = 1, 2, \ldots n$$

## * Axiomatic probability :-

P(A) is the probability function defined on a σ-Field B of events if the following properties or axioms hold.

① $P(A) \geq 0$

② $P(S) = 1$

③ $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

## * Note :-

$$P(S) = \frac{n(S)}{n(S)} = 1$$

**Theorem:-** For two mutually exclusive (or disjoint) events A and B defined on sample space S, show that -

$$P(A \cup B) = P(A) + P(B)$$

**Answer:-** We know that,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

But, here A and B are mutually exclusive i.e. disjoint events.

$$\therefore n(A \cap B) = 0$$

$$\therefore n(A \cup B) = n(A) + n(B)$$

Divide both side by $n(S)$

$$\therefore \frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B)}{n(S)}$$

$$\therefore \frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)}$$

$$\therefore \boxed{P(A \cup B) = P(A) + P(B)}$$

**Theorem:-** prove that, probability of impossible event is zero i.e. $P(\phi) = 0$

**Answer:-** Impossible event contains no sample point and hence the certain event S and the impossible event $\phi$ are mutually exclusive.

$$\therefore S \cup \phi = S \Rightarrow P(S \cup \phi) = P(S)$$

$$\therefore P(S) + P(\phi) = P(S)$$

$$\therefore P(\phi) = P(S) - P(S) = 0$$

$$\therefore \boxed{P(\phi) = 0}$$

**Theorem :-** prove that,

$$P(\bar{A}) = 1 - P(A)$$

For any event A and complementary event $\bar{A}$ on sample space S.

**proof :-**

Let A and $\bar{A}$ are mutually disjoint events

$\therefore A \cup \bar{A} = S$

$\therefore$ Take probability on both side

$\therefore P(A \cup \bar{A}) = P(S)$

$\therefore P(A) + P(\bar{A}) = P(S)$ ------ $\left\{ P\left(\bigcup_{i=1}^{n} P(A)\right) = \sum_{i=1}^{n} P(A) \right\}$

But $P(S) = 1$

$\therefore P(A) + P(\bar{A}) = 1$

$$\therefore \boxed{P(\bar{A}) = 1 - P(A)}$$

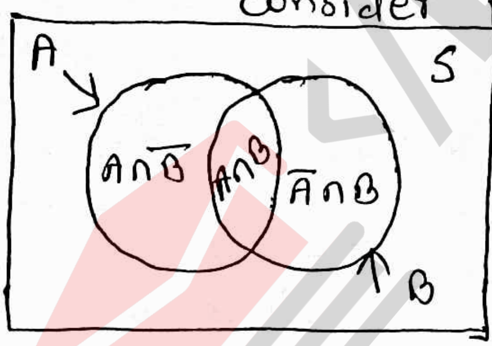
* **Theorem :-** For any two events A and B, prove that -

① $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

② $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

**proof :-**

consider following venn diagram




① From venn diagram

$B = (A \cap B) \cup (\bar{A} \cap B)$

where $(A \cap B)$ and $(\bar{A} \cap B)$ are disjoint events

$\therefore$ Take probability on both sides

$\therefore P(B) = P(A \cap B) + P(\bar{A} \cap B)$

$$\therefore \boxed{P(\bar{A} \cap B) = P(B) - P(A \cap B)}$$

② Similarly, $A = (A \cap B) \cup (A \cap \overline{B})$
where $(A \cap B)$ and $(A \cap \overline{B})$ are disjoint events.
Take probability on both sides.

$$\therefore \quad P(A) = P(A \cap B) + P(A \cap \overline{B})$$

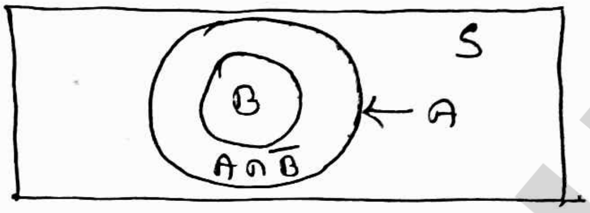
$$\therefore \quad \boxed{P(A \cap \overline{B}) = P(A) - P(A \cap B)}$$

* <u>Theorem:-</u> If $B \subset A$ then prove that,

① $P(A \cap \overline{B}) = P(A) - P(B)$

② $P(B) \leq P(A)$

<u>Proof:-</u> consider the following venn diagram




① Since $B \subset A$.
From venn diagram $B$ and $A \cap \overline{B}$ are mutually exclusive (disjoint) events.

$$\therefore \quad A = B \cup (A \cap \overline{B})$$

Take probability on both side

$$\therefore \quad P(A) = P(B) + P(A \cap \overline{B})$$

$$\therefore \quad \boxed{P(A \cap \overline{B}) = P(A) - P(B)}$$

② consider,

$$P(A \cap \overline{B}) = P(A) - P(B)$$

Let $P(A \cap \overline{B}) \geq 0$

$$\Rightarrow \quad P(A) - P(B) \geq 0$$

$$\Rightarrow \quad P(A) \geq P(B)$$

$$\Rightarrow \quad \boxed{P(B) \leq P(A)}$$

* **Theorem:-** If A and B are any two events (subsets of sample space S) and are not disjoint, then prove that -

$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

**Proof:-** Since A and B are not disjoint events.

We know that,

$$n(A\cup B) = n(A)+n(B)-n(A\cap B)$$

Divide both side by $n(S)$

$$\therefore \frac{n(A\cup B)}{n(S)} = \frac{n(A)+n(B)-n(A\cap B)}{n(S)}$$

$$\therefore \frac{n(A\cup B)}{n(S)} = \frac{n(A)}{n(S)}+\frac{n(B)}{n(S)}-\frac{n(A\cap B)}{n(S)}$$

$$\therefore \boxed{P(A\cup B) = P(A)+P(B)-P(A\cap B)}$$

* **Theorem:-** For three non-mutually exclusive events A, B and C prove that -

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A) + P(A\cap B\cap C)$$

**proof:-** consider,

$$P(A\cup B\cup C) = P[A\cup(B\cup C)]$$

$$= P(A)+P(B\cup C)-P[A\cap(B\cup C)]$$

$$= P(A)+[P(B)+P(C)-P(B\cap C)]-P[(A\cap B)\cup(A\cap C)]$$

$$= P(A)+P(B)+P(C)-P(B\cap C)-[(A\cap B)+P(A\cap C) - P(A\cap B\cap C)]$$

$$= P(A)+P(B)+P(C)-P(B\cap C)-P(A\cap B)-P(A\cap C)+P(A\cap B\cap C)$$

$$\boxed{\therefore P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)}$$

Que] A letter of the English alphabet is choosen at random. calculate the probability that the letter so choosen -

(i) is a vowel

(ii) precedes m and is a vowel

(iii) Follows m and is a vowel

Answer:- There are 26 letters in english alphabet. The sample space of the experiment is

$S = \{a, b, c, ------, x, y, z\}$

$\therefore \boxed{n(S) = 26}$

(i) Let $E_1$ be the event that the letter choosen is a vowel.

$\therefore E_1 = \{a, e, i, o, u\}$

$\therefore n(E_1) = 5$

$\therefore \boxed{P(E_1) = \frac{n(E_1)}{n(S)} = \frac{5}{26}}$

(ii) Let $E_2$ be the event that the letter precedes m and is a vowel.

$\therefore E_2 = \{a, e, i\}$

$\therefore n(E_2) = 3$

$\therefore \boxed{P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{26}}$

(iii) Let $E_3$ be the event that the letter Follows m and is a vowel.

$\therefore E_3 = \{o, u\} \Rightarrow n(E_3) = 2$

$\therefore P(E_3) = \frac{n(E_3)}{n(S)} = \frac{2}{26}$

$\therefore \boxed{P(E_3) = \frac{1}{13}}$

Que] Five salesman of A, B, C, D and E of a company are considered for a three-member trade delegation to represent the company in an international trade conference.

construct the sample space and find the probability that -

① A is selected

② A is not selected

③ Either A or B (noth both) is selected

Answer:-

The sample space for selecting three salesman out of 5 salesman A, B, C, D and E for the trade delegation is given by -

$$S = \{ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE\}$$

$$\therefore \boxed{n(S) = 10}$$

① Let $E_1$ is event that A is selected

$$\therefore E_1 = \{ABC, ABD, ABE, ACD, ACE, ADE\}$$

$$\therefore n(E_1) = 6$$

$$\therefore \boxed{P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{10} = \frac{3}{5}}$$

② Let E2 is even that A is not selected

means $E_2 = \overline{E_1}$

$\therefore$

$$\therefore \boxed{P(\overline{E_1}) = 1 - P(E_1) = 1 - \frac{3}{5} = \frac{2}{5}}$$

③ Let $E_3$ is event that either A or B (not both) is selected

$$\therefore E_3 = \{ACD, ACE, BCD, BCE, BDE, ADE\}$$

$$\therefore n(E_3) = 6$$

$$\therefore \boxed{P(E_3) = \frac{n(E_3)}{n(S)} = \frac{6}{10} = \frac{3}{5}}$$

Que] A, B and C are three mutually exclusive and exhaustic events associated with a random experiment. Find P(A), P(B), P(C) given that -

$$P(B) = \frac{3}{2} P(A) \quad \text{and} \quad P(C) = \frac{1}{2} P(B)$$

Answer:- It is given that,

$$P(B) = \frac{3}{2} P(A) \rightarrow (1)$$

$$P(C) = \frac{1}{2} P(B) \rightarrow (2)$$

Let $P(A) = p$

$\therefore (1) \Rightarrow P(B) = \frac{3}{2} P(A) = \frac{3}{2} p = \frac{3p}{2}$

$(2) \Rightarrow P(C) = \frac{1}{2} P(B) = \frac{1}{2} \times \frac{3p}{2} = \frac{3p}{4}$

Since A, B and C are mutually exclusive and exhaustic events.

$\therefore P(A) + P(B) + P(C) = 1$

$\therefore p + \frac{3p}{2} + \frac{3p}{4} = 1$

$\therefore \frac{4p + 6p + 3p}{4} = 1$

$\therefore \frac{13p}{4} = 1$

$\therefore \boxed{p = \frac{4}{13}}$

$\therefore \boxed{P(A) = p = \frac{4}{13}}$

$\boxed{P(B) = \frac{3}{2} P(A) = \frac{3}{2} \times \frac{4}{13} = \frac{6}{13}}$

$\boxed{P(C) = \frac{1}{2} P(B) = \frac{1}{2} \times \frac{6}{13} = \frac{3}{13}}$

Que] The probability that a student passes a physics test is $\frac{2}{3}$ and the probability that he passes both a physics and an English test is $\frac{14}{15}$. The probability that he passes atleast one test is $\frac{4}{5}$. What is the probability that he passes the English test ?

Answer:- Let us define the following events

A → The student passes a physics test

B → The student passes an English test

So It is given that,

$P(A) = \frac{2}{3}$, $P(A \cap B) = \frac{14}{15}$, $P(A \cup B) = \frac{4}{5}$.

We find $P(B)$.

We know that,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore \frac{4}{5} = \frac{2}{3} + P(B) - \frac{14}{15}$$

$$\therefore \frac{4}{5} + \frac{14}{15} - \frac{2}{3} = P(B)$$

$$\therefore \frac{12+14-10}{15} = P(B)$$

$$\therefore \frac{16}{15} = P(B)$$

$$\therefore \boxed{P(B) = \frac{16}{15}}$$

Que] An MBA applies for a job in two firms X and Y. The probability of his being selected in firm X is 0.7 and being rejected at Y is 0.5. The probability of at least one of his applications being rejected is 0.6. What is probability that he will be selected in one of the firms ?.

Answer:-

Let A denote the event that the person is selected in firm X.

$\therefore$ $\overline{A}$ denote the event that the person is rejected at firm X.

B denote the event that the person is selected in firm Y.

$\therefore$ $\overline{B}$ denote the event that the person is rejected at firm Y.

It is given that,

The probability of his being selected in firm X is 0.7.

$\therefore$ $P(A) = 0.7 \rightarrow (1)$

$\therefore$ $P(\overline{A}) = 1 - P(A) = 1 - 0.7 = 0.3 \rightarrow (2)$

The probability of his being rejected at firm Y is 0.5.

$\therefore$ $P(\overline{B}) = 0.5 \rightarrow (3)$

$\therefore$ $P(B) = 1 - P(\overline{B}) = 1 - 0.5 = 0.5 \rightarrow (4)$

The probability of atleast one of his applications being rejected is 0.6

$\therefore$ $P(\overline{A} \cup \overline{B}) = 0.6$

$\therefore$

But $P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$

$\therefore \quad 0.6 = 0.3 + 0.5 - P(\bar{A} \cap \bar{B})$

$\therefore \quad 0.6 = 0.8 - P(\bar{A} \cap \bar{B})$

$\therefore \quad P(\bar{A} \cap \bar{B}) = 0.8 - 0.6 = 0.2$

$\therefore$ The probability that the person will be selected in one of the two firms x or y is

$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$

$= 1 - 0.2$

$= \underline{\underline{0.8}}$

Que] Three newspapers A, B and C are published in a certain city. It is estimated from a survey that of the adult population — 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read both B and C, 2% read all three. Find what percentage read atleast one of the newspaper ?.

Answer:- Let define following events

$E_1$: The adult read newspaper A.

$E_2$: The adult read newspaper B.

$E_3$: The adult read newspaper C.

It is given that

$P(E_1) = 20\% = \frac{20}{100}$

$P(E_2) = 16\% = \frac{16}{100}$

$P(E_3) = 14\% = \frac{14}{100}$

$P(E_1 \cap E_2) = 8\% = \frac{8}{100}$

$P(E_1 \cap E_3) = 5\% = \frac{5}{100}$

$P(E_2 \cap E_3) = 4\% = \frac{4}{100}$

$P(E_1 \cap E_2 \cap E_3) = 2\% = \frac{2}{100}$

∴ The required probability that an adult reads atleast one of the newspaper is

$$P(E1 \cup E2 \cup E3) = P(E1) + P(E2) + P(E3) - P(E1 \cap E2) - P(E2 \cap E3) - P(E1 \cap E3) + P(E1 \cap E2 \cap E3)$$

$$= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{4}{100} - \frac{5}{100} + \frac{2}{100} = \frac{35}{100} = 0.35$$

Hence, ~~0.35~~

35% of the adult population reads atleast one of the newspapers.