2D TRANSFORMATIONS

Transformations

- **Rigid Body Transformations -** transformations that do not change the object.
- Translate
  - If you translate a rectangle, it is still a rectangle
- Scale
  - If you scale a rectangle, it is still a rectangle
- Rotate
  - If you rotate a rectangle, it is still a rectangle

Translation

- A translation moves all points in an object along the same straight-line path to new positions.
- The path is represented by a vector, called the translation or shift vector.
- We can write the components:

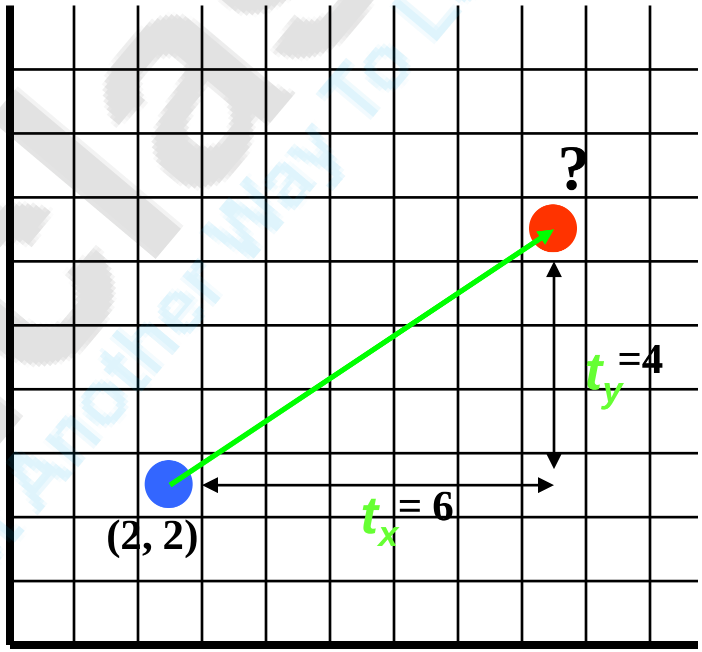
$$x' = x + t_x$$

$$y' = y + t_y$$

- or in matrix form:

$$P' = P + T$$

$$\begin{pmatrix} x' & y' \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} + \begin{pmatrix} t_x & t_y \end{pmatrix}$$



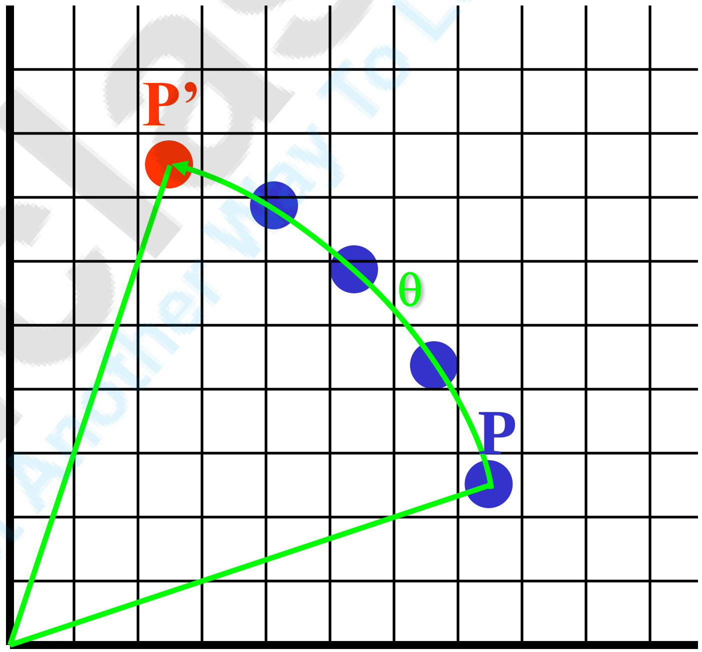
Example:

Q. Translate the polygon with coordinate A(2,5) , B(7,10) and C(10,2) by 3 units in x direction and 4 units in y direction.

Q. Translate the polygon with coordinate A(12,15) , B(-17,10) and C(10,21) by 5 units in x direction and 7 units in y direction .

Rotation

- Rotation - Repositions an object along a circular path
- Rotation requires an Θ and a pivot point
- First, we'll assume the pivot is at the origin.



Rotation

- Review Trigonometry

=> $\cos\phi = x/r$, $\sin\phi = y/r$

- $x = r.\cos\phi$, $y = r.\sin\phi$

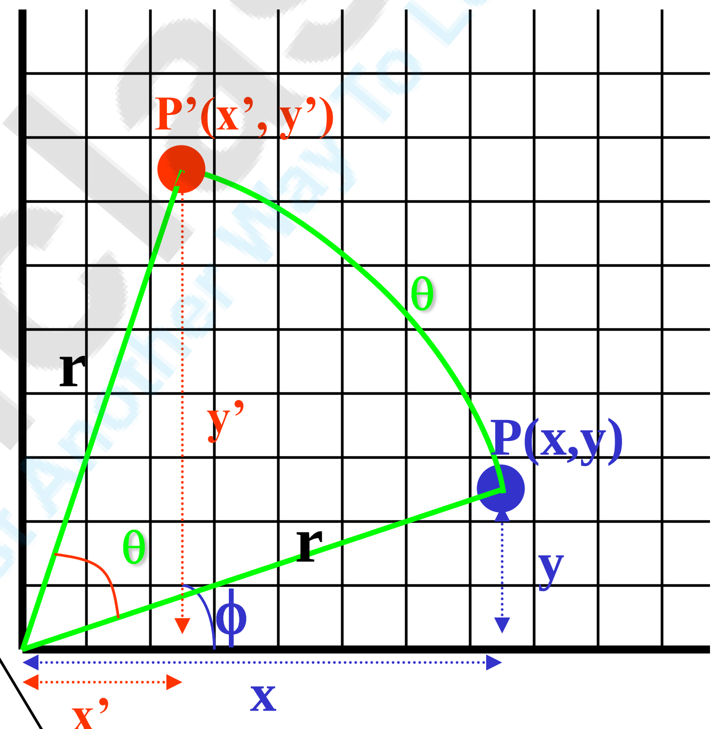
=> $\cos(\phi + \theta) = x'/r$

- $x' = r.\cos(\phi + \theta)$
- $x' = r.\cos\phi\cos\theta - r.\sin\phi\sin\theta$
- $x' = x.\cos\theta - y.\sin\theta$

=> $\sin(\phi + \theta) = y'/r$

$y' = r.\sin(\phi + \theta)$

- $y' = r.\cos\phi\sin\theta + r.\sin\phi\cos\theta$
- $y' = x.\sin\theta + y.\cos\theta$



Identity of Trigonometry

Rotation

- **We can write the components:**

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

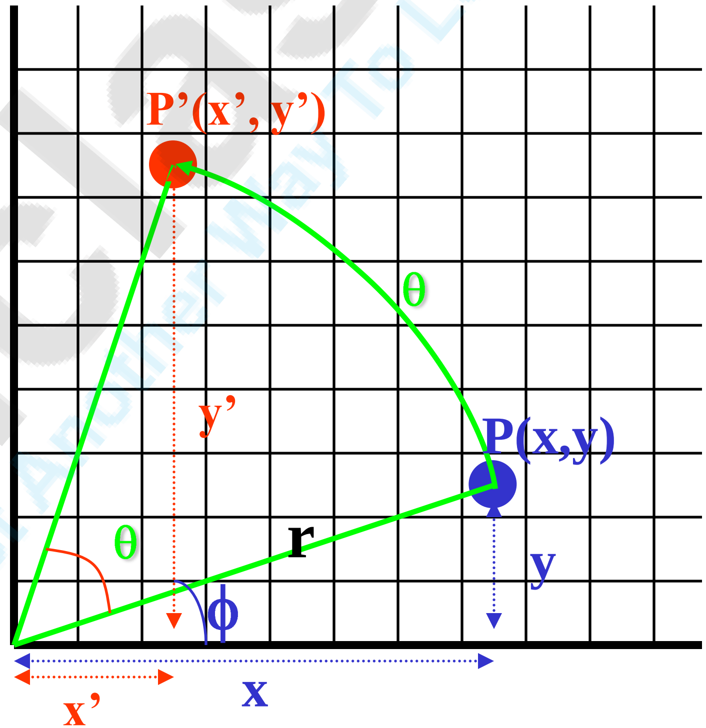
- **or in matrix form:**

$$P' = R \bullet P$$

- **θ can be clockwise (-ve) or counterclockwise (+ve as our example).**
- **Rotation matrix**

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

- **Default direction of rotation is anticlockwise direction.**



$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{pmatrix} \cos \theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

example

- A point (4,3) is rotated counterclockwise by an angle of 45° . Find the rotation matrix and resultant point.

Scaling

- **Scaling changes the size of an object and involves two scale factors, S_x and S_y for the x- and y- coordinates respectively.**
- **Scales are about the origin.**
- **We can write the components:**

$$x' = s_x \bullet p_x$$

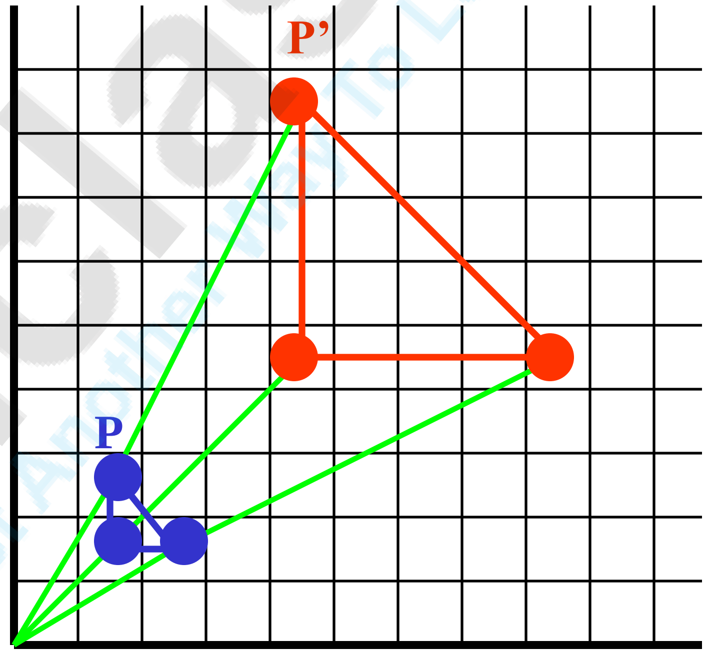
$$Y' = s_y \bullet p_y$$

or in matrix form:

$$P' = S \bullet P$$

Scale matrix as:

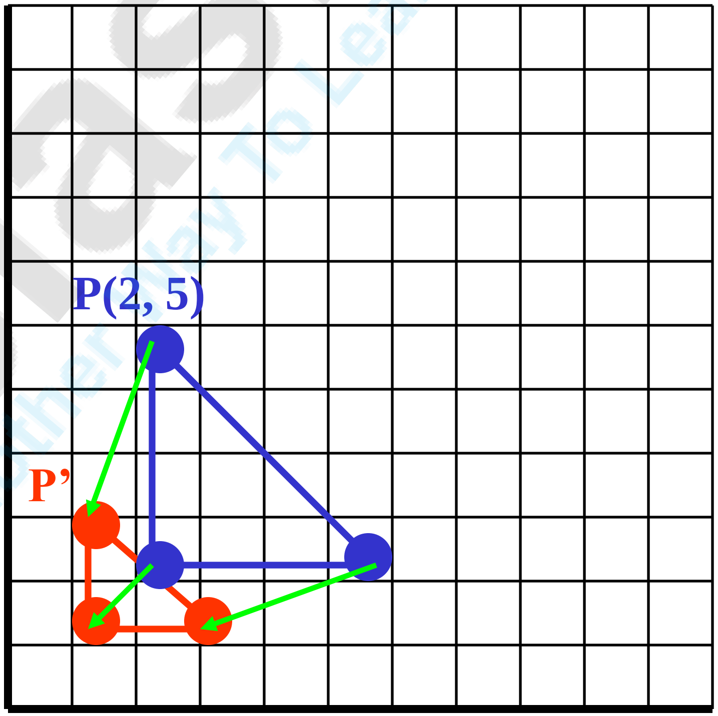
$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



Scaling

- If the scale factors are in between 0 and 1 ➔ the points will be moved closer to the origin ➔ the object will be smaller.

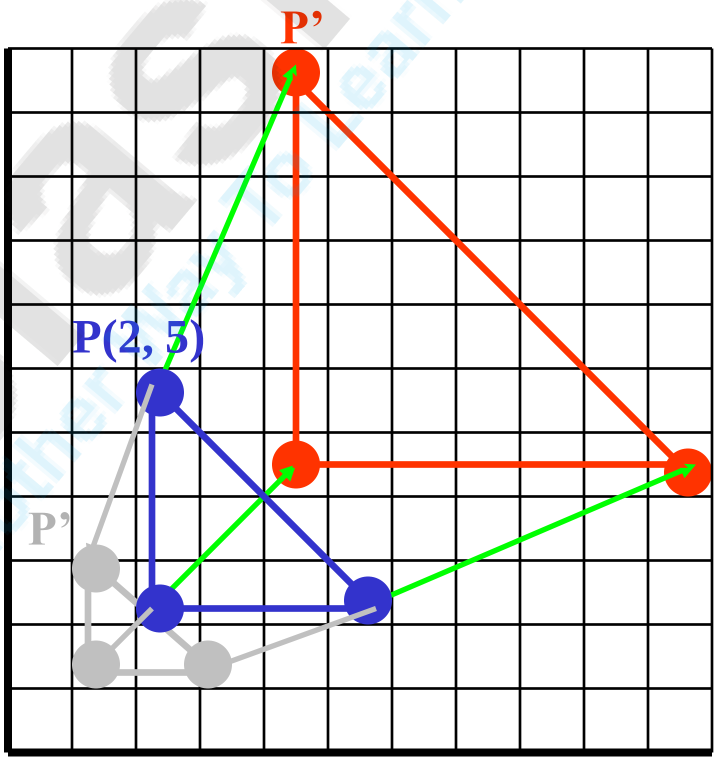
- Example :
  - P(2, 5), Sx = 0.5, Sy = 0.5
  - Find P' ?



Scaling

- If the scale factors are in between 0 and 1 ➔ the points will be moved closer to the origin ➔ the object will be smaller.
- Example :
  - P(2, 5), Sx = 0.5, Sy = 0.5
  - Find P' ?

- If the scale factors are larger than 1 ➔ the points will be moved away from the origin ➔ the object will be larger.

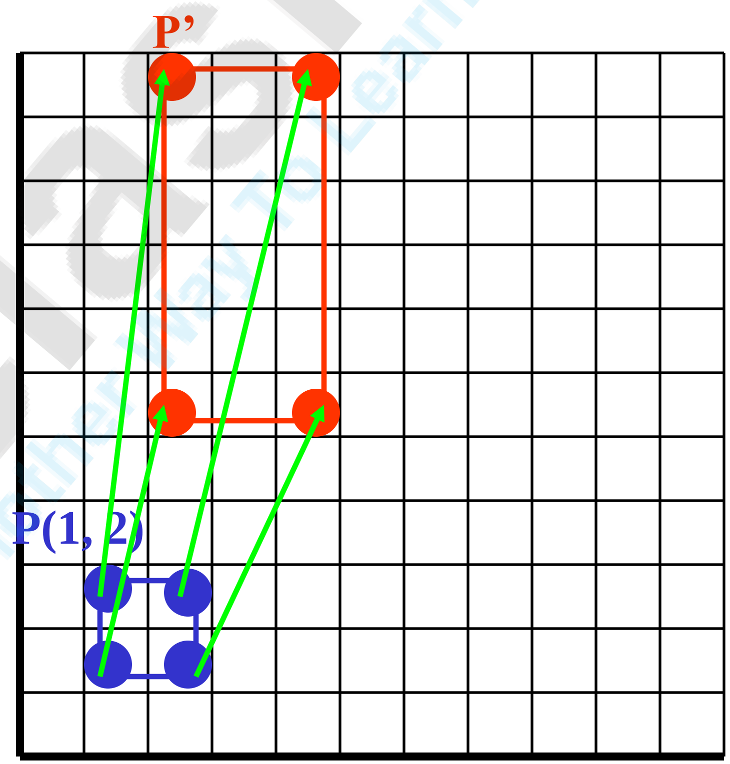


- Example :
  - P(2, 5), Sx = 2, Sy = 2
  - Find P' ?

Scaling

- If the scale factors are the same, $S_x = S_y$ ➔ uniform scaling
- Only change in size (as previous example)

  - If $S_x \neq S_y$ ➔ differential scaling.
  - Change in size and shape
  - Example : square → rectangle
    - P(1, 3), $S_x = 2$, $S_y = 5$, P' ?



Homogeneous coordinates

- To fit the picture in to proper position many time we required to perform sequence of transformation i.e. translation , rotation and scaling.
- We have a general transformation of a point:

$$P' = P\,M1 + M2$$

- When we scale or rotate, we set M1, and M2 is the additive identity. i. e. zero matrix
- When we translate, we set M2, and M1 is the multiplicative identity .i.e. identity matrix
- To combine multiple transformations, we must explicitly compute each transformed point.

Homogeneous coordinates

- In order to combine sequence of transformation we have to eliminate the matrix addition associated with translation term in M2.
- To achieve this we have to represent M1 as 3X3 matrix instead of 2 X 2 by introducing dummy coordinate called Homogeneous coordinate
- Homogeneous coordinate is represented by the triplet (X_w , Y_w W)

  Where

  $X = X_w / W$, $Y = Y_w / W$

- For 2D transformation W can be any positive value but we take W= 1
- Each 2D position can be represented ith homogeneous coordinate as

  (x , y , 1)

Homogeneous coordinate for translation are:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{pmatrix}$$

$$\begin{pmatrix} X' & Y' & 1 \end{pmatrix} = \begin{pmatrix} X & Y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{pmatrix}$$

$$= \begin{pmatrix} X+t_x & Y+t_y & 1 \end{pmatrix}$$

Homogeneous coordinate for rotation are:

$$R = \begin{pmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} X' & Y' & 1 \end{pmatrix} = \begin{pmatrix} X & Y & 1 \end{pmatrix} \begin{pmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x\cos\Theta - y\sin\Theta & x\sin\Theta + y\cos\Theta & 1 \end{pmatrix}$$

Homogeneous coordinate for scaling are:

$$S = \begin{pmatrix} S_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} X' & Y' & 1 \end{pmatrix} = \begin{pmatrix} X & Y & 1 \end{pmatrix} \begin{pmatrix} S_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x\,s_x & y\,s_y & 1 \end{pmatrix}$$

Q. Give 3 x 3 Homogeneous coordinate transformation matrix for each of the following translation :

1. **Shift the image right to 3 unit**
2. **Shift the image up 2 units**
3. **Move the image down ½ unit and right 1 unit**
4. **Move the image down 2/3 unit and left 4 units**

Q. Find transformed matrix that transform the given square ABCD to half its size with center till remaining at the same position . The coordinates of the square are A(1,1) , B (3 , 1) C (3 , 3) D (1, 3) and center at (2 ,2) also find resultant coordinates of square.

Solution:

1. **Translate the square so that its center coincides with origin**
2. **Scale the square with respective the origin**
3. **Translate the square back to the original position.**

tx = ty = 2   sx =sy = 0.5

Q. find transformation of triangle A(1,0) B(0 , 1) C(1,1) by

1. Rotating 45° about the origin and then translating 1 unit in x and y direction
2. Translating 1 unit in x and y direction and then rotating 45° about the origin

Composite Transformation

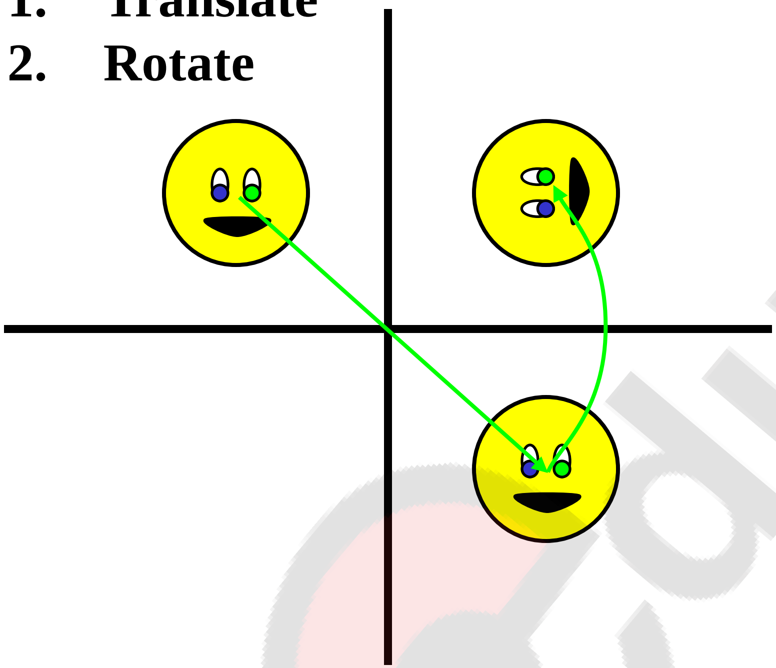
- We can represent any sequence of transformations as a single matrix.

- Composite transformations:
  - Rotate about an arbitrary point – translate, rotate, translate
  - Scale about an arbitrary point – translate, scale, translate
  - Change coordinate systems – translate, rotate, scale

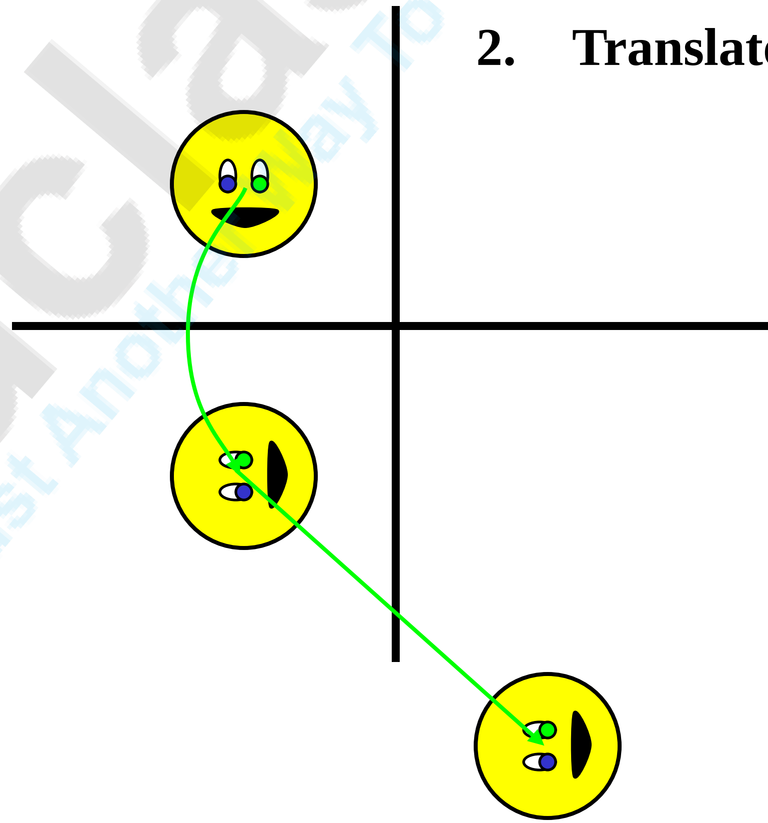
Order of operations

So, it does matter. Let's look at an example:

1. **Translate**
2. **Rotate**



1. **Rotate**
2. **Translate**



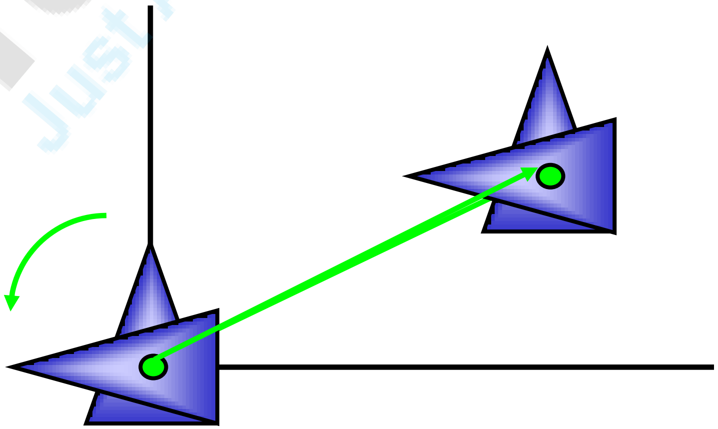
Composite Transformation Matrix

General Pivot- Point Rotation

Operation :-

1. Translate (pivot point is moved to origin)
2. Rotate about origin
3. Translate (pivot point is returned to original position)

T(pivot) • R(θ) • T(–pivot)



Composite Transformation Matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -tx & -ty & 1 \end{pmatrix} \bullet \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx & ty & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -tx\cos\theta + ty\sin\theta & -tx\,_{\sin}\theta - ty\,_{\cos}\theta & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx & ty & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -tx\cos\theta + ty\sin\theta + tx & -tx\,_{\sin}\theta - ty\,_{\cos}\theta + ty & 1 \end{pmatrix}$$

Q. Perform counter clockwise 45° rotation of triangle A(2,3) B(5,5) C(4,3) about point (1 , 1)

Composite Transformation Matrix

- Example
  - Perform 60° rotation of a point P(2, 5) about a pivot point (1,2). Find P'?

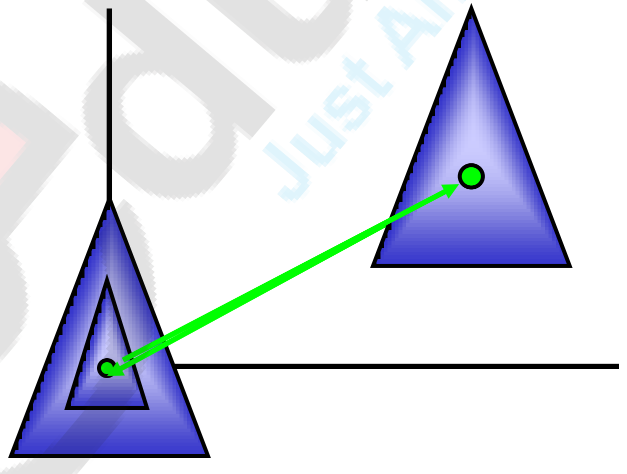
Composite Transformation Matrix

General Fixed-Point Scaling

Operation :-

1. **Translate (fixed point is moved to origin)**
2. **Scale with respect to origin**
3. **Translate (fixed point is returned to original position)**

T(fixed) • S(scale) • T(–fixed)



- Find the matrix that represents scaling of an object with respect to any fixed point?

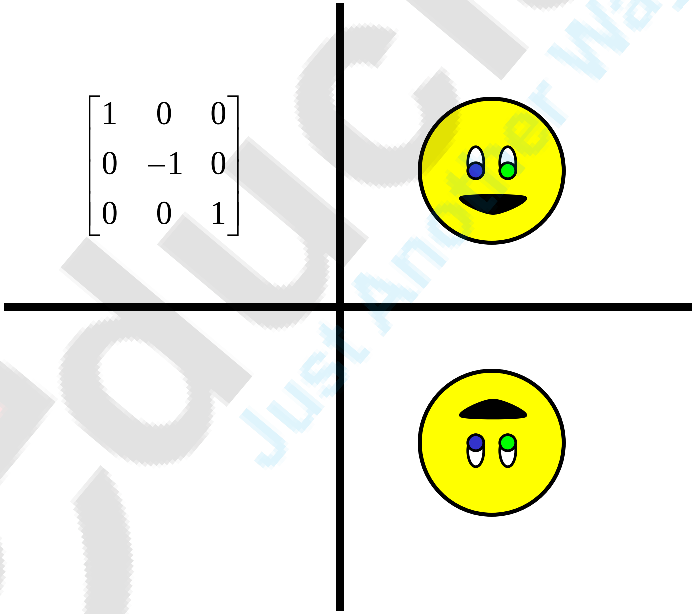
- Given P(6, 8) , Sx = 2, Sy = 3 and fixed point (2, 2). Use that matrix to find P'?

Other transformations

Reflection: It is a transformation that produce mirror image of an object relative to an axis of reflection

x-axis

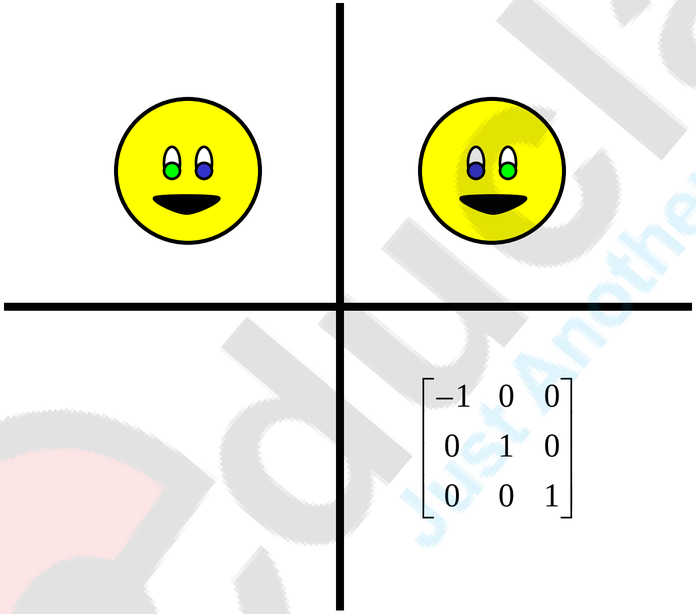
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Other transformations

Reflection: about y axis

y-axis

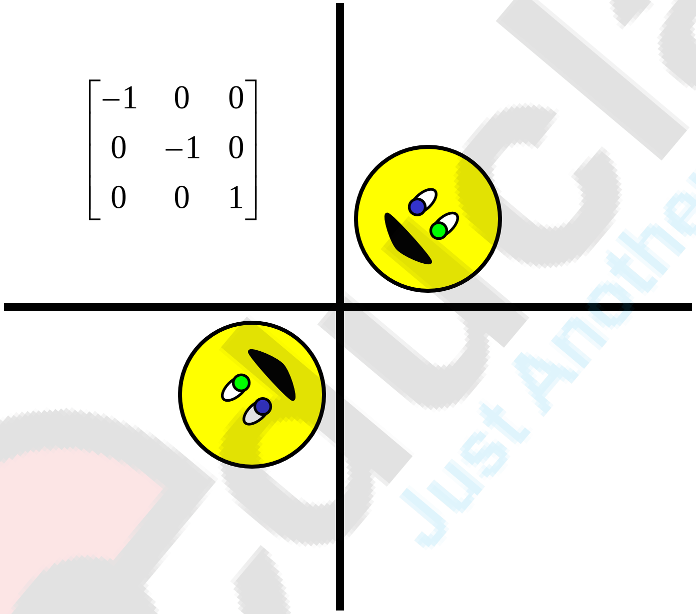


$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Other transformations

Reflection: about origin

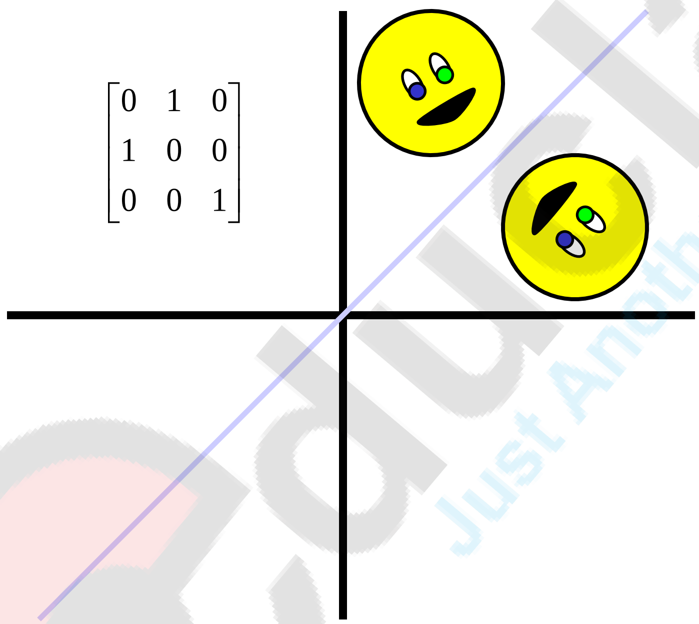
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Other transformations

Reflection: about y=x

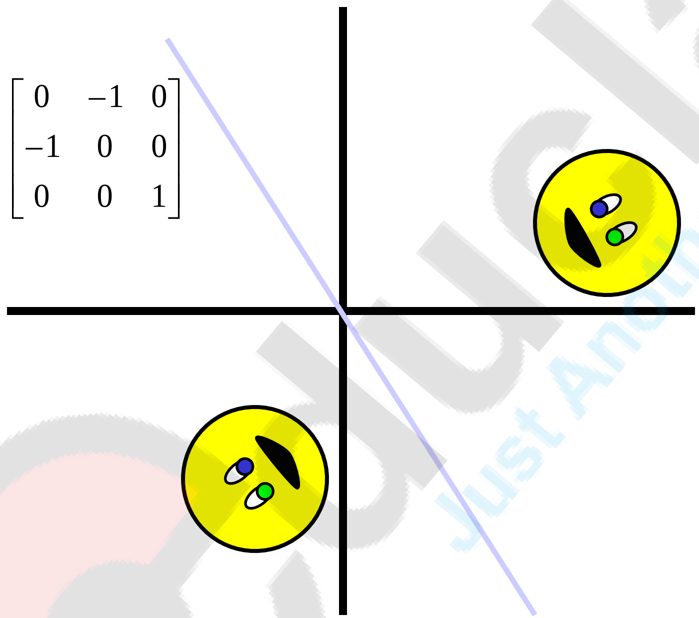
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Other transformations

Reflection: about y= -x

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

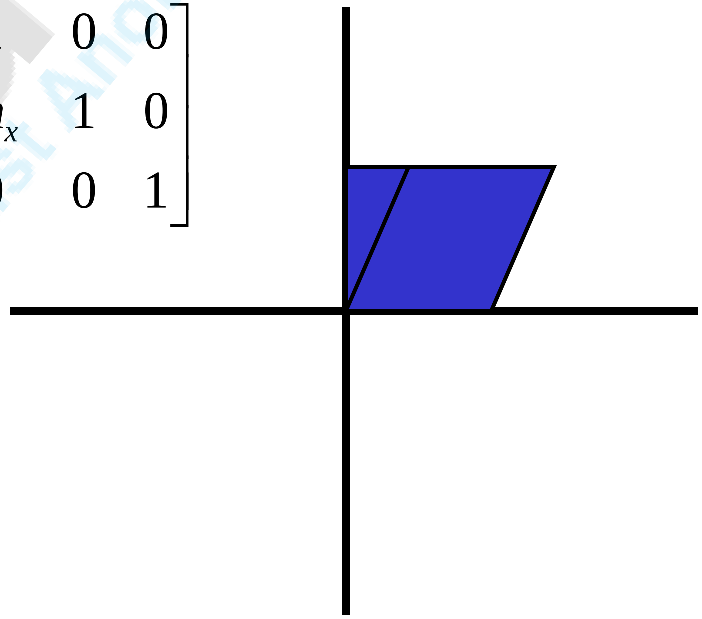


Other transformations

Shear : Transformation that slant the shape of an object .

X shear: preserve the y coordinate but change x value which causes vertical line to tilt right

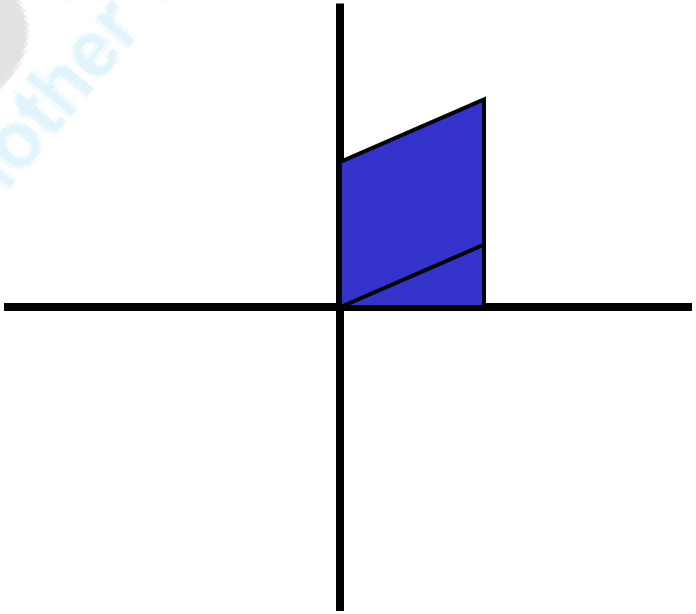
$$\begin{bmatrix} 1 & 0 & 0 \\ sh_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Other transformations

y shear: preserve the x coordinate but change y value which causes horizontal line to transform in to lines which slope up or down

$$\begin{bmatrix} 1 & sh_y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Inverse Transformation

- When we apply any transformation to point P(x,y), we get a new point (x’ , y’).
- Sometimes it may require to get undo the applied transformation.
- In this case we have to get original point (x , y) from (x’ , y’).
- This can be achieved by inverse transformation

- If the inverse of matrix T is T^{-1} then ,

  $TT^{-1}=T^{-1}T=I$, Identity matrix
- The elements of inverse matrix can be calculated from the elements of T as ,

$$t_{ij}^{-1} = [\ (-1)^{i+j} \det M_{ij}\ ] / \det T$$

Determinant of a 2 x 2 matrix is ;

$$\det \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix} = t_{11}\, t_{22} - t_{12}\, t_{21}$$

- Inverse of homogeneous coordinate transformation matrix can be given as ,

$$\begin{pmatrix} a & d & 0 \\ b & e & 0 \\ c & f & 1 \end{pmatrix}^{-1} = \frac{1}{ae-bd} \begin{pmatrix} e & -d & 0 \\ -b & a & 0 \\ bf-ce & cd-af & ae-bd \end{pmatrix}$$