P_1 and W_1 with shipping cost 3. The column total (i.e. market requirement) corresponding to this cell is 20 while the row total (Plant capacity) is 30. So we allocate 20 units to this cell. Not the market requirement of W_1 has been satisfied, we move horizontally in the row P_1 to the cell $P_1 W_2$.

2. The market requirements of column W_2 is 20 while a total of 10 units are left in row P_1. Thus, 10 units are assigned to this cell. With this, the supply of the row P_1 is exhausted.

3. Now, move vertically to the cell P_2 W_2. For this cell, the market requirement remains 10 and the plant capacity being 25, assign 10 units to this cell and exhaust the requirement of market W_2. Then, move to the cell $P_2 W_3$ and allocate remaining 15 units of plant P_2 to this cell, exhausting the capacity of plant P_2 leaving requirement of 20 units for market W_3.

4. Now capacity of plant P_3 is 20 units which will satisfy the requirement of market W_3 so we allocate 20 units to cell $P_3 W_3$ exhausting all plant capacities and market requirements.

These assignments are shown in the table below :

Plants	Markets W_1	Markets W_2	Markets W_3	Plant capacities
P_1	20 / 3	10 / 4	2	30 / 10 / 0
P_2	2	10 / 1	15 / 5	25 / 15 / 0
P_3	4	3	20 / 3	20 / 0
Market requirements	20 / 0	20 / 10 / 0	35 / 20 / 0	

2. The Least-Cost Method

This method involves following steps for finding initial feasible solution :

i) The least cost method starts by making the first allocation to that cell whose shipping cost per unit is lowest.

ii) This lowest cost cell is loaded or filled as much as possible in view of the plant capacity of its row and the market requirements of its column. In our example, the cell $P_2 W_2$ has the least shipping cost, so we allocate 20 units to this cell exhausting the requirements of market W_2.

iii)We move to the next lowest cost cell and make an allocation in view of the remaining capacity and requirement of its row and column. In case there is a tie for the lowest cost cell during any allocation we make an allocation in the cell which will satisfy either the maximum market requirement or exhaust the plant capacity. In this example, there is a tie between cells $P_1 W_3$ and the cell $P_2 W_1$. So we choose cell $P_1 W_3$ because it will exhaust maximum capacity i.e. 30 units.

iv)The above procedure is repeated till all rim requirements are satisfied.

The initial feasible solution of the given problem using the least-cost method is given in the following table.

Plants	Markets W_1	W_2	W_3	Plant capacities
P_1	3	4	30 2	30/0
P_2	5 2	20 1	5	25/5/0
P_3	15 4	3	5 3	20/15/0
Market requirements	20/15/0	20/0	35/5/0	

3. Vogel's Approximation Method

Various steps involved for finding a initial feasible solution are given below:

i) For each row of the transportation matrix, calculate difference between the smallest and the next smallest element. Similarly for each column of the matrix, compute the difference of the smallest and next smallest element of the column.

ii) Identify the row or column with largest difference. If a tie occurs, use a row or column which will either exhaust the maximum plant capacity or satisfy the maximum market requirements. In the given example, the largest difference is 2, thus the column designated as W_2 is selected.

iii) Allocate the maximum feasible quantity in the box corresponding to the smallest cost in that row or column. Eliminate that row or column where an allocation is made. The lowest cost in column W_2 is 1, hence the cell P_2 W_2 is selected for allocation, 20 units are assigned to this cell and column W_2 is deleted as the requirement of market W_2 is satisfied.

iv) Re-compute row and column differences for the reduced transportation table. The above procedure is repeated until all the rim requirements are satisfied. The assignments made by VAM are given in the following table.

Plants	Markets			Plant capacity	Diff.
	W_1	W_2	W_3		
P_1	3	4	30 2	30/0	1/1/1
P_2	5 2	20 1	5	25/5/0	1/3
P_3	15 4	3	5 3	20/15/0	0/1/1
Market requirements	20/15/0	20/0	35/5/0		
Diff.	1/1/1	2	1/1/1		

The minimum total transportation cost associated with this solution is

Total cost = (3 × 8 + 4 × 4 + 6 × 1 + 4 × 4 + 6 × 2 + 6 × 3) × 10 = Rs. 920

6.4 CHECK YOUR PROGRESS

Determine an initial basic feasible solution to the following transportation problems using (a) north-west corner method and (b) Vogel's method.

1.

		Destination				
		E	E	F	G	Supply
Source	A	11	13	17	14	250
	B	16	18	14	10	300
	C	21	24	13	10	400
	Demand	200	225	275	250	

2. Consider the transportation problem having the following cost and requirements table :

	Destination					
		1	2	3	4	Supply
	1	5	8	3	6	30
Source	2	4	5	7	4	50
	3	6	2	4	5	40
	Demand	30	20	40	30	

6.5 STEPS OF MODI METHOD (TRANSPORTATION ALGORITHM)

The steps to evaluate unoccupied cells are as follows :

Step 1 For an initial basic feasible solution with m + n -1 occupied cells, calculate u_i and v_j for rows and columns. The initial solution can be obtained by any of the three methods discussed earlier.

To start with, any one of u_j's or v_j's is assigned the value zero. It is better to assign zero for a particular u_i or v_j where there are maximum number of allocation in a row or column respectively, as it will reduce arithmetic work considerably. Then complete the calculation of u_i's and v_j's for other rows and columns by using the relation

$$c_{ij} = u_i + v_j \text{ for all occupied cells (i, j).}$$

Step 2 For unoccupied cells, calculate opportunity cost (the difference that indicates the per unit cost reduction that can be achieved by an allocation in the unoccupied cell) by using the relationship

$$d_{ij} = c_{ij} - (u_i + v_j) \text{ for all i and j.}$$

Step 3 Examine sign of each d_{ij}

i) If $d_{ij} > 0$, then current basic feasible solution is optimal.

ii) If $d_{ij} = 0$, then current basic feasible solution will remain unaffected but an alternative solution exists.

iii) If one or more $d_{ij} < 0$, then an improved solution can be obtained by entering unoccupied cell (i, j) in the basis. An occupied cell having the largest negative value of d_{ij} is chosen for entering into the solution mix (new transportation schedule).

Step 4 Construct a closed-path (or loop) for the unoccupied cell with largest negative opportunity cost. Start the closed path with the selected unoccupied cell and mark a plus sign (+) in this cell, trace a path along the rows (or column) to an occupied cell, mark the corner with minus sign (-) and continue down the column (or row) to an occupied cell and mark the

corner with plus (+) sign and minus sign (-) alternatively. Close the path back to the selected unoccupied cell.

Step 5 Select the smallest quantity amongst the cell marked with minus sign on the corners of closed loop. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus sign and subtract it from the occupied cells marked with minus sign.

Step 6 Obtain a new improved solution by allocating units to the unoccupied cell according to Step 5 and calculate the new total transportation cost.

Step 7 Test the revised solution further for optimality. The procedure terminates when all $d_{ij} \geq 0$, for unoccupied cells.

Remarks :

1. The closed-loop (path) starts and ends at the selected unoccupied cell. It consists of successive horizontal and vertical (connected) lines whose end points must be occupied cells, except for an end point associated with entering unoccupied cell. This means that every corner element of the loop must be an occupied cell.

   It is immaterial whether the loop is traced in a clockwise or anti-clockwise direction and starting up, down, right or left (but never diagonally). However, for a given solution only one loop can be constructed for each unoccupied cell.
2. There can be only one plus (+) sign and only one minus (-) sign in any row or column.

3. The closed path indicates changes involved in reallocating the shipments.

6.6 DEGENERACY AND ITS RESOLUTION

A basic feasible solution for the general transportation problem must consist of exactly m + n -1 (number of rows + number of columns - 1) positive allocations in independent positions in the transportation table. A solution will be called degenerate when the number of occupied cells is less than the required number, m + n - 1. In such cases, the current solution cannot be improved because it is not possible to draw a closed path for every occupied cell. Also, the values of dual variables u_i and v_j which are used to test the optimality cannot be computed. Thus, we need to remove the degeneracy to improve the given solution. The degeneracy in the transportation problems may occur at two stages :

a) When obtaining an initial basic feasible solution we may have less than m + n - 1 allocations.

b) At any stage while moving towards optimal solution. This happens when two or more occupied cells with the same minimum allocation become unoccupied simultaneously.

Case 1 Degeneracy at the initial solution :

To resolve degeneracy at the initial solution, we proceed by allocating a very small quantity close to zero to one or more (if needed) unoccupied cells so as to get m + n -1 number of occupied cells. This amount is denoted by a Greek letter ε (epsilon) or Δ (delta). This quantity would not affect the total cost as well as supply and demand values. In a minimization transportation problem it is better to allocate Δ to unoccupied cells that have lowest transportation costs, whereas in maximization problems it should be allocated to a cell that has a high payoff value. In some cases, Δ must be added in one of those unoccupied cells which makes possible the determination of u_i and v_j uniquely.

The quantity Δ is considered to be so small that if it is transferred to an occupied cell it does not change the quantity of allocation. That is,

$$x_{ij} + \Delta = x_{ij} - \Delta = x_{ij}$$

$$\Delta - \Delta = 0; \qquad \Delta + \Delta = \Delta$$

$$0 + \Delta = \Delta; \qquad k \times \Delta = \Delta$$

It is also then obvious that Δ does not affect the total transportation cost of the allocation. Hence, the quantity Δ is used to evaluate unoccupied cells and to reduce the number of improvement cycles necessary to reach an optimal solution. Once purpose is over, Δ can be removed from the transportation table.

The minimum total transportation cost associated with this solution is

Total cost = (3 × 8 + 4 × 4 + 6 × 1 + 4 × 4 + 6 × 2 + 6 × 3) × 10 = Rs. 920

Cost 2 Degeneracy at subsequent iterations :

To resolve degeneracy which occurs during optimality test, the quantity may be allocated to one or more cells which have become unoccupied recently to have m + n - 1 number of occupied cells in the new solution.

6.7 ADDITIONAL PROBLEM

Example 6.3

A manufacturer wants to ship 22 loads on his product as shown below. The matrix gives the kilometers from sources of supply to the destinations.

		Destination					
		D_1	D_2	D_3	D_4	D_5	Supply
Source	S_1	5	8	6	6	3	8
	S_2	4	7	7	6	5	5
	S_3	8	4	6	6	4	9
	Demand	4	4	5	4	8	22 / 25

Shipping cost is Rs. 0 per load per km. What shipping schedule should be used to minimize total transportation cost?

Solution :

Since the total destination requirement of 25 units exceeds the total resource capacity of 22 by 3 units, the problem is unbalanced. The excess requirement is handled by adding a dummy plant, S_{excess} with a capacity equal to 3 units. We use zero unit transportation cost to the dummy plant. The modified transportation table is shown in Table 6.3.1.

Initial solution 6.3.1

	D_1	D_2	D_3	D_4	D_5	Supply	
S_1	5	8	6 (5)	6	3 (3)	8	$u_1 = 0$
S_2	4 (4)	7	7	6 (1)	5	5	$u_2 =$
S_3	8	4 (4)	6	6	4 (5)	9	$u_3 = 1$
S_{excess}	0	0	0	0 3	0	3	← $u_4 =$
Demand	4	4	5	4	8	25	
	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	$v_5 =$		

The initial solution is obtained by using Vogel's approximation method as shown in Table 6.3.1. Since the solution includes 7 occupied cells, therefore, the initial solution is degenerate. In order to remove degeneracy we assign Δ to unoccupied cell (S_2, D_5) which has minimum cost among unoccupied cells as shown in Table 6.3.2.

Table 6.3.2

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	5 +3	8 +5	6 5 (-)	6 +2	3 3 (+)	8	$u_1 = 0$
S_2	4 4	7 +2	7 -1	6 1 (+)	5 Δ (-)	5	$u_2 = 2$
S_3	8 +5	4 4	6 -1	6 +1	4 5	9	$u_3 = 1$
S_{excess}	0 +2	0 +1	0 (+) -2	0 3 (-)	0 -7	3	$u_4 = -4$
Demand	4	4	5	4	8	25	
v_j	$v_1 = 2$	$v_2 = 3$	$v_3 = 6$	$v_4 = 4$	$v_5 = 3$		

Determine u_i and v_j for occupied cells as shown in Table 6.3.2. Since opportunity cost in the cell (S_{excess}, D_3) is largest negative, it must enter the basis and the cell (S_2, D_5) must leave the basis. The new solution is shown in Table 6.3.3.

Table 6.3.3

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	5 +3	8 +5	6 5 (-)	6 +2	3 3 (+)	8	$u_1 = 0$
S_2	4 4	7 +4	7 +1	6 1	5 +2	5	$u_2 = 0$
S_3	8 +3	4 4	6 -1	6 ( ) + -1	4 5 (-)	9	$u_3 = 1$
S_{excess}	0 +2	0 +3	0 Δ (+)	0 3 (-)	0 +3	3	$u_4 = -6$
Demand	4	4	5	4	8	25	
v_j	$v_1 = 4$	$v_2 = 3$	$v_3 = 6$	$v_4 = 6$	$v_5 = 3$		

Repeat the procedure of testing optimality of the solutions given in Table 6.33. The optimal solution is shown in Table 6.3. 4

Table 6.3. 4

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	5	8	6	6	3 8	8
S_2	4 4	7	7	6 1	5	5
S_3	8	4 4	6 2	6 3	4	9
S_{excess}	0	0	0 3	6	0	3
Demand	4	4	5	4	8	25

The minimum total transportation cost associated with this solution is

Total cost = (3 × 8 + 4 × 4 + 6 × 1 + 4 × 4 + 6 × 2 + 6 × 3) × 10 = Rs. 920

Example 6.4

Solve the following transportation problem for minimum cost:

Destinations	A	Origins B	C	D	Requirements
1	7	4	3	4	15
2	3	2	7	5	25
3	4	4	3	7	20
4	9	7	5	3	40
Availabilities	12	8	35	25	

Solution

Since this is an unbalanced transportation problem, introduce a dummy activity E with availability of 20 units. Apply vogel's method to find initial feasible solution.

	A	B	C	D	E	Requirement	1	2	3	4	5	6
1	7	4	15 3	4	0	15 0	3	1	1	1	1	1
2	12 3	8 2	7	5 5	0	25 17 12	2	1	2	2	2	2
3	4	4	20 3	7	0	20 0	3	1	1	1	4	-
4	9	7	5	20 3	20 0	40 20 0	3	2	2	-	-	-

Availability	*12* / *0*	*8* / *0*	*35* / *15* / 0	*25* / *5* / *0*	*20* / *0*	100

Differene	1	2	0	1	0
	1	2	0	1	0
	1	-	0	1	-
	1	-	0	1	-
	-	-	0	1	-
	-		4	1	-

The initial feasible solution is given below :

	A	B	C	D	E	Requirements
1	7	4	15 3	4	0	15
2	12 3	8 2	7	5 5	0	25
3	4	4	20 3	7	0	20
4	9	7	5	20 3	20 0	40
Availability	12	8	35	25	20	

This solution is tested for optimality. Since there are 7 allocations we place a small entity (e) in the cell (E, 1). Assuming $u_1 = 0, u_i$ and v_j are computed below :

	A	B	C	D	E	u_i
1	7	4	15 3	4	e 0	0
2	12 3	8 2	7	5 5	0	2
3	4	4	20 3	7	0	0
4	9	7	5	20 3	20 0	0
v_j	1	0	3	3	0	

6	4		1	
		2		-2
3	4		4	0
8	7	2		

l_{ij} matrix

Since one of the l_{ij} is -ve including the most-negative marginal cost cell (E,2) in the loop and test the solution for optimality.

7	4	15	4	e 0
12 3	8 2	7	5 - θ 5	+ θ 0
4	4	20 3	7	0
9	7	5	20 + θ 3	20 - θ 0

$$\theta = \min\{(5-\theta, 20-\theta)\}$$
$$\theta = 5$$

Assume u_i 0 u_i *and* v_j are calculated below : u_i

	7	4	15 3	4	0 e 0
	12 3	8 2	7	5	5 0 0
	4	4	20 3	7	0 0
v_j	9 3	7 2	5 3	25 3 3	15 0 0

4	2		1	
		4	2	
1	2		4	0
6	5	2		

$_{ij}$ matrix

Since all $_{ij} \geq 0$, this is optimal solution and the allocation is given by

FROM	C	A	B	E	C	D	E	
TO	1	2	2	2	3	4	4	
QTY.	15	12	8	5	20	25	15	
COST	45	36	16	0	60	75	0	= 232 Total Cost

Example 6.5

Consider the following transportation cost table. The costs are given in Rupees, the supply and demand are in units. Determine an optimal solution.

Destination / Source	*1*	*2*	*3*	*4*	*5*	*Supply*
I	*40*	*36*	*26*	*38*	*30*	*160*
II	*38*	*28*	*34*	*34*	*198*	*280*
III	*36*	*38*	*24*	*28*	*30*	*240*
Demand	*160*	*160*	*200*	*120*	*240*	

Solution

Since demand is greater than supply by 200 units, we introduce a dummy source with a supply of 200 units. We, then, find initial solution by vogel's method.

Destination

Source	1	2	3	4	5		Difference				
I	40	36	26	38	160 30	160/0	4	4	4	4	4
II	38	160 28	120 34	34	198	280/120/0	6	6	6	0	164
III	36	38	80 24	120 28	40 30	240/120/0	4	4	4	4	6
Dummy	160 0	0	0	0	40 0	200/40/0	0	0			

Demand	160/0	160/0	200/80/0	120/0	240/200/40
Difference	36	28 28 8	24 24 2 2	28 28 6 6	30 30 0 0

Thus the initial solution is given by the table given above. To test whether the initial solution is optimal, we apply optimality test. If the solution is not optimal, then we will try to improve the solution to make it optimal. To test the optimality of the initial solution, we find whether the number of allocations is equal to m + n – 1 or not. Also, these allocations should be independent.

Let us introduce the variables u_i and v_j. To determine the values of these variables, let $u_3 = 0$, the values of other variables are shown in the following table.

	1	2	3	4	5	u_i
I	40	36	26	38	160 30	0
II	38	16 28	120 34	34	198	10
III	36	38	80 24	120 28	40 30	0
Dummy	16 0	0	0	0	40 0	-30
v_j	30	18	24	28	30	

Let us now find $_{ij} \quad C_{ij} \quad (u_i \quad v_j)$ for the non basic cells. 1 2

	3	4	5	u_i	
I	10 18	2	10 160	0	
II	-2 16	120 - θ	- 4 + θ 158	10	θ = 120
III	6 20	80 + θ	120 - θ 40	0	
Dummy	16 12	6	2 40	-30	
v_j	30 18	24	28 30		

Since there are negative $_{ij}$ *s* this solution is not optimal. Including the most negative $_{ij}$ in the loop, the next improved solution is given below.

We test this solution for optimality. Since the total number of basic cells is less than *8 (= m + n – 1)*, we put *e* which is a very small quantity in the least cost independent cell and compute values of $_i$ and v_j

Destinations

Source	1	2	3	4	5	α_i
I	40	36	26	38	160 30	30
II	38	16 28	34	120 34	198	34
III	36	38	200 24	28	40 30	30
Dummy	16 0	0	0	e 0	40 0	0
v_j	0	-6	-6	0	0	

The values of $\alpha_{ij} = C_{ij} - (u_i + v_j)$ are given in the table below :

	1	2	3	4	5	α_i
I	10	12	2	8	160	30
II	4	16	6	120	164	34
III	6	14	200	-2 + θ	40 - θ	30
Dummy	16	6	6	e - θ	40 + θ	0
v_j	0	-6	-6	0	0	

Since there is one negative α_{ij} this solution is not optimal. Including this negative value in the loop, the next improved solution is given below which is tested for optimality.

Destinations

Source	1	2	3	4	5	α_i
I	40	36	26	38	160 30	0
II	38	16 28	34	120 34	198	6
III	36	38	200 24	e 28	40 30	0
Dummy	16 0	0	0	0	40 0	- 30
v_j	30	22	24	28	30	

Now we find Δ_{ij} for non basic cells

Destination

sources	1	2	3	4	5	α_i
I	10	14	2	10	160	0
II	2	16	4	120	162	6
III	6	16	200	e	40	0
Dummy	16	8	6	2	40	-30
v_j	30	22	24	28	30	

Since all Δ_{ij} s are positive, the above table will give optimal solution. The optimal solution is given below.

From Source	*To Destination*	*No. of Units*	*Cost per unit Rs.*
I	*5*	*160*	*30*
II	*2*	*160*	*28*
II	*4*	*120*	*34*
III	*3*	*200*	*24*
III	*5*	*40*	*30*

and the associated total cost is given by

160 × Rs. 30 + 160 × Rs. 28 + 120 × Rs. 34 + 200 × Rs. 24 + 40 × Rs. 30 = Rs. 19,360.

Example 6.6

Solve the following transportation Problem :

GODOWNS

		1	*2*	*3*	*4*	*5*	*6*	*Stock Available*
FACTORY	*1*	*7*	*5*	*7*	*7*	*5*	*3*	*60*
	2	*9*	*11*	*6*	*11*	-	*5*	*20*
	3	*11*	*10*	*6*	*2*	*2*	*8*	*90*
	4	*9*	*10*	*9*	*6*	*9*	*12*	*50*
DEMAND		*60*	*20*	*40*	*20*	*40*	*40*	

Note : it is not possible to transport any quantity from factory, 2 to Godown 5. State whether the solution derived by you is unique.

Solution

The initial solution is found by VAM below :

Factory	Godowns						Availability	Diff.
	1	2	3	4	5	6		
1	7	20 5	7	7	5	40 3	60/40/0	2/4/0
2	10 9	11	10 6	11	∞	5	20/10/0	1/3
3	11	10	30 6	20 2	40 2	8	90/70/30/0	0/4/2/5
4	50 9	10	9	6	9	12	50/0	3/0
Demand	60 ~~50~~ 0	20 0	40 ~~10~~ 0	20 0	40 0	40 0		
Diff.	2	5	0/1	4	3	2		

The initial solution is tested for optimality. Since there are only 8 allocations and we require 9 (m + n – 1 = 9) allocations, we put a small quantity e in the least cost independent cell (2,6) and apply the optimality test. Let $u_3 = 0$ and then we calculate remaining u_i and v_j

	1	2	3	4	5	6	u_i
1	7	20 5	7	7	5	40 3	-2
2	10 9	11	10 6	11		e 5	0
3	11	10	30 6	20 2	40 2	8	0
4	50 9	10	9	6	9	12	0
v_j	9	7	6	2	2	5	

Now we calculate $\Delta_{ij} = C_{ij} - u_i - v_j$ for non basic cells which given in the table below :

0		3	7	5	
	4		9		
2	3				3
	3	3	4	7	7

Δ_{ij}

Since all Δ_{ij} are positive, the initial solution found by VAM is an optimal solution. The final locations are given below :

Factory	*to Godown*	*Unit*	*Cost*	*Value*
1	2	20	5	100
1	6	40	3	120
2	1	10	9	90
2	3	10	6	60
3	3	30	6	180
3	4	20	2	40
3	5	40	2	80
4	1	50	9	450
		Total cost Rs.		1,120

The above solution is not unique because the opportunity cost of cell (1,2) is zero. Hence alternate solution exists. Students may find that the alternate solution is as given below :

Factory	*to Godown*	*Unit*	*Cost*	*Value*
1	1	10	7	70
1	2	20	5	100
1	6	30	3	90
2	3	10	6	60
2	6	10	5	50
3	3	30	6	180
3	5	40	2	80
3	4	20	2	40
4	1	50	9	450
			Total cost Rs.	1,120

Example 6.7

STRONGHOLD Construction company is interested in taking loans from banks for some of its projects P, Q, R, S, T. The rates of interest and the lending capacity differ from bank to bank. All these projects are to be completed. The relevant details are provided in the following table. Assuming the role of a consultant, advise this company as to how it should tae the loans so to the total interest payable will be the

least. Are there alternate optimum solutions? If so, indicate one such solution.

	Interest rate in percentage for PROJECT					Max. Credits (in thousands)
Bank	P	Q	R	S	T	
Pvt. Bank	20	18	18	17	17	Any amount
Nationalized Bank	16	16	16	15	16	400
Co-operative Bank	15	15	15	13	14	250
Amount required (in thousands)	200	150	200	125	75	

Solution

The total amount required by five projects is Rs. 750 thousands. Since private bank can give credit for any amount, we allocate [Rs. 750 – (Rs. 400 + Rs. 250) = Rs. 100] thousand to private banks. The balanced problem is given below. The initial solution is found using VAM.

	P	Q	R	S	T	Max. Cred.	Diff.
Pvt. Bank	20	10 (18)	18	17	17	100/0	0/1/0/0
Nationalised Bank	20 (16)	50 (16)	15 (16)	15	16	400/200/50/0	1/0/0/0
Co-operative Bank	15	15	5 (15)	125 (13)	7 (14)	250/125/50/0	1/1/0
amount	200	150	200	125	75		
required	0	100	150	0	0		
		0	0				
difference	1	1	1	2	2		
	1	1	1	-	2		
	1	1	1	-	-		
	4	2	2	-	-		

Let us test initial solutions for optimality. There are m + n – 1 = 7 independent allocations.

Let us now introduce $u_i, v_j, i = 1,2,3$ and $j = 1,2,...5$.

Let $u_2 = 0$ and calculate remaining u_i $i = 1,3$ and v_j's $j = 1,2,...5$

Calculate $_{ij} = C_{ij} - u_i - v_j$ for non allocated cells.

	P	Q	R	S	T	u_i
Pvt. Bank	2 20	10 18	0 18	1 17	0 17	2
Nationalised Bank	200 16	50 16	15 16	1 15	1 16	0
Co-operative Bank	0 15	0 15	50 15	12 13	75 14	-1
v_j	16	16	16	14	15	

Since none of the Δ_{ij} is negative, the initial solution obtained as above is optimal one.

Total interest (as per above allocation)

= 100 × .18 + 200 × .16 + 50 × .16 + 150 × .16 + 50 × .15 + 125 × .13 + 75 × .14

= Rs 116.25 thousands

= Rs. 1,16,250

Further, since some of the Δ_{ij} 's are zero, therefore the above solution is not unique. To find out an alternative solution, let us include the cell (CB,Q) as the basic cell so the new solution is given below :

	P	Q	R	S	T
Pvt. Bank	20	10 18	18	17	17
Nationalised Bank	200 16	16	15 16	15	16
Co-operative Bank	15	15	50 15	12 13	75 14

Since there are only 6 allocations, let us introduce e-a small quantity in the least independent cell (NB,S). We now introduce $u_i, i \square 1,2,3$ and $v_i, j \square 1,2,...5$. Let $u_3 \square u$ Calculate $\lceil_{ij} \square C_{ij} \square u_i \square v_j$ for non basic cells.

	P	Q	R	S	T	u_i
Pvt. Bank	3 20	10 18	1 18	1 17	0 17	3
Nationalised Bank	200 16	-1 +θ 16	20 16	e -θ 15	0 16	2
Co-operative Bank	1 15	50 -θ 15	1 15	12 +θ 13	75 14	0
v_j	14	15	14	13	14	

Since $\lceil_{ij}$ for cell (NB,Q) is negative, this solution is not optimal. We include the cell (NB,Q) in the basic cell

$0 = [\min (50 - \theta, e - \theta) = \theta] = e$

So the new solution given below is tested for optimality.

	P	Q	R	S	T	u_i
Pvt. Bank	2 20	10 18	0 18	1 17	0 17	2
Nationalised Bank	200 16	e 16	20 16	1 15	1 16	0
Co-operative Bank	0 15	50 15	0 15	12 13	75 14	-1
v_j	16	16	16	14	15	

Since all $\lceil_{ij}$ are non-negative, the above solution is optimal one and the total cost is

= 100 × 018 + 200 × .16 + 200 × .16 + 50 × .15 + 125 × .13 + 75 × .14 = Rs. 116.25 thousands = Rs. 1,16,250

Note : The alternative solution can be found by taking any cell with zero $\lceil_{ij}$ as the basic cell.

Example 6.8

Solve the following transportation problem to maximize profit and give criteria for optimality :

	Profit (Rs.) / Unit destination				
Origin					*Supply*
	1	*2*	*3*	*4*	
A	*40*	*25*	*22*	*33*	*100*
B	*44*	*35*	*30*	*30*	*30*
C	*38*	*38*	*28*	*30*	*70*
Demands	*40*	*20*	*60*	*30*	

Solution

As the given matrix is a profit matrix and the objective is to maximize profit, we first of all convert the profit matrix into a loss matrix for solving it by transportation method. This is done by multiplying the given matrix by minus sign to get the following matrix.

Destination		*Profit (Rs.) / Unit destination*				
Origin						*Supply*
		1	*2*	*3*	*4*	
A		*-40*	*-25*	*-22*	*-33*	*100*
B		*-44*	*-35*	*-30*	*-30*	*30*
C		*-38*	*-38*	*-28*	*-30*	*70*
Demands		*40*	*20*	*60*	*30*	

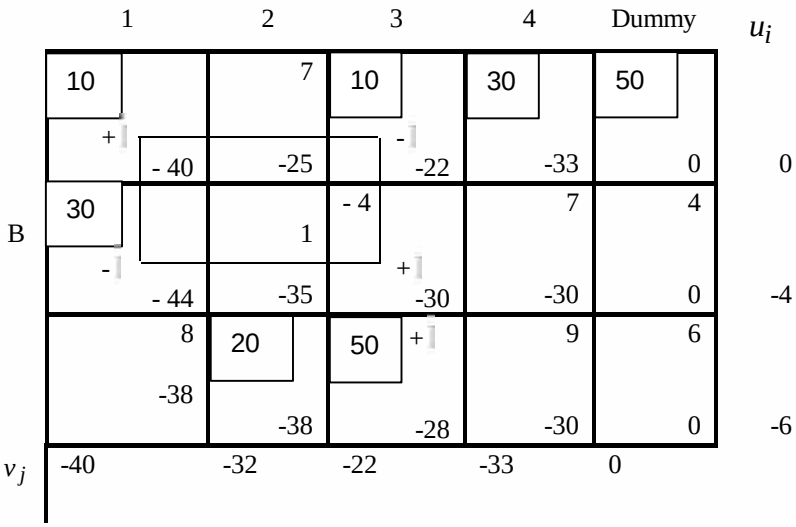
Further, the above loss matrix is not a balanced one as (supply > demand), we will therefore introduce a dummy destination and find the initial feasible solution using VAM.

	1	2	3	4	Dummy	Supply	Diff.
A	- 40	-25	20 -22	30 -33	50 0	100/70/50/0	7/7/7/1
B	30 - 44	-35	-30	-30	0	30/0	9
C	10 -38	20 -38	40 -28	-30	0	70/50/40/0	0/0/8/2
Demands	40/10/0	20/0	60/20/0	30/0	50/0		
Diff.	4/2/2	3/13	2/6/6/6	3/3	0/0/0/0		

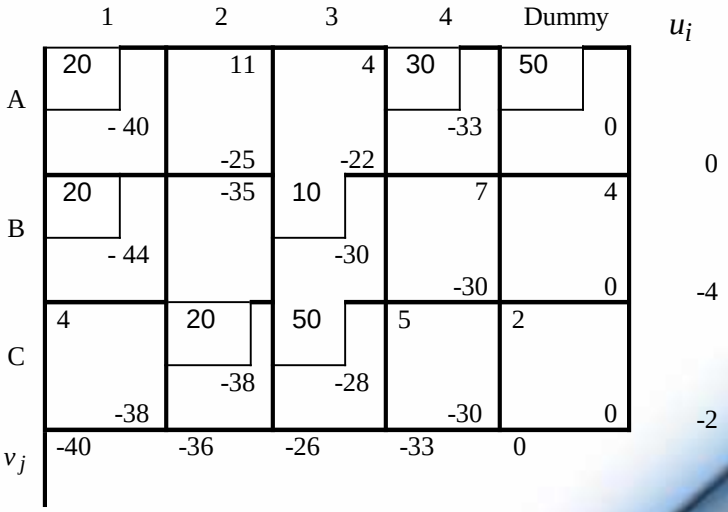
The initial feasible solution after introducing the variables u_i and v_j is tested for optimality as below :

	1	2	3	4	Dummy	u_i
A	- 8 - 40	20 -25	30 - -22	50 -33	0	0
	30					
	- 44	-35	4 -30	15 -30	12 0	-12
	10 -	20 -38	40 -38	+ -28	9 -30	6 0	-6
Demands	40/10/0	20/0	60/20/0	30/0			

Since some of the $_{ij} \quad C_{ij} \quad (u_i \quad v_i)$ for non-allocated cells shown in the above table are not positive, the solution obtained above is not an optimal solution. In order to obtain an optimal solution, place a small assignment θ in the cell with most negative value of $_{ij}$. In the above table, is placed in the most negative cell (A,1) and a loop is formed including this cell. Now, the maximum value which can take is 10 units. After putting this value of in the cell (A,1), other allocation cells in the loop will also get affected as shown in the following table.



Calculate again $_{ij} \quad C_{ij} \quad (u_i \quad v_i)$ after determining the value of u_i and v_j for non-allocated cells. Cell (B,3) has most negative value in the above table. We place a small allocation and determine its value as described above which comes out to be 10 units. Putting the value of θ, we get the following table.



In the above table, we find that the values of $_{ij} \quad C_{ij} \quad (u_i \quad v_i)$ are positive for all non-allocated cells. Hence we can say that allocation shown by the table are optimal. The final allocation after converting the loss matrix into profit matrix are given below :

	1	2	3	4	Dummy
A	20 40	25	22	30 33	50 0
B	20 44	35	10 30	30	0
C	38	20 38	50 28	30	0

and the profit is given by

Profit = 20 × Rs. 40 + 30 × Rs. 33 + 20 × Rs. 44 + 10 × Rs. 30 + 20 × Rs. 38 + 50 × Rs. 28

= Rs. 800 + Rs. 990 + Rs. 880 + Rs. 300 + Rs. 760 + Rs. 1,400

= Rs. 5,130.

Note : The above problem can also be converted into a minimization problem by subtracting each element of the given matrix from 44 (the highest element).

Example 6.9

A manufacturer of jeans is interested in developing an adverting campaign that will reach four different age groups. Advertising campaigns can be conducted through TV, Radio and Magazines. The following table gives the estimated cost in paise per exposure for each age group according to the medium employed. In addition, maximum exposure levels possible in each of the media, namely TV, Radio and Magazines are 40, 30 and 20 millions respectively. Also the minimum desired exposures within each age group, namely 13-18, 19-25, 26-35, 36 and older, are 30,25,15 and 10 millions. The objective is to minimize the cost of attaining the minimum exposure level in each age group.

Media	*Age Groups*			
	13-18	*19-25*	*26-35*	*36 & older*
TV	*12*	*7*	*10*	*10*
Radio	*10*	*9*	*12*	*10*
Magazines	*14*	*12*	*9*	*12*

i) Formulate the above as a transportation problem, and find the optimal solution.

ii) Solve this problem if the policy is to provide at least 4 million exposures through TV in the 13-18 age group and at least 8 million exposures through TV in the age group 19-25

Solution

i) As a first step, let us formulate the given problem as a transportation problem :

Media	Age Groups				
	13-18	19-25	26-35	36 & older	Exposures available (in million)
TV	12	7	10	10	40
Radio	10	9	12	10	30
Magazines	14	12	9	12	20
Minimum number of exposures required (in millions)	30	25	15	10	80/90

It is apparent from the above that this transportation problem is an unbalanced on it is balanced by introducing a dummy category before applying Vogel's Approximation method.

Media	13-18	19-25	26-35	36 & older	Dummy category	Max. Exp. (in million)	Penalty
TV	12	2 (7)	5 (10)	10 (10)	0	40/15/0	7/3/0/0/0
Radio	30 (10)	9	12	10	0	30/0	9/1/0/0
Magazines	14	12	10 (9)	12	1 (0)	20/10/0	9/3/3
Min. no. of Exposures required (million)	30/0	25/0	15/5/0	10/0	10/0	90	
Penalty	2	2	1	0	0		
	2	2	1	0	-		
	2	-	1	0	-		
	2	-	2	0	-		
	-	-	10	10	-		

The total cost for this allocation works out to Rs. 71.5 lakhs

The solution given by VAM is degenerate since there are only six assignments. Let us put an E in the least cost independent cell to check for optimality. Let $u_1 = 0$ and we calculate remaining u_i and v_js .

Media	13-18	19-25	26-35	36 & older	Dummy category	u_i
TV	12	25 (7)	5 (10)	10 (10)	0	0
Radio	30 (10)	9	12	10	E (0)	-1
Magazines	14	12	1 (9)	12	10 (0)	-1
v_j	11	7	10	10	1	

Let us not calculate $_{ij} = C_{ij} - u_i - v_j$ for empty cells.

		$_{ij}$ S		
1				-1
	3	3	1	
4	6		3	

Since one of the $_{ij}$ s is negative, the solution given above is not optimal. Let us include the cell with negative $_{ij}$ as a basic cell and try to improve the solution. The reallocated solution is given below which is tested for optimality.

	13-18	19-25	26-35	36 & older	Dummy category	u_i
TV	12	25 (7)	10	10 (10)	5 (0)	0
Radio	30 (10)	9	12	10	E (0)	0
Magazines	14	12	1 (9)	12	5 (0)	0
v_j	10	7	9	10	0	

Let us calculate Δ_{ij} for empty cells.

2		1		
	2	3	0	
4	5		2	

Since all the entries in the last table are non-negative, the second solution is optimal.

Through TV, 25 million people must be reached in the age group 19-25 and 10 million people in the age group 36 & older.

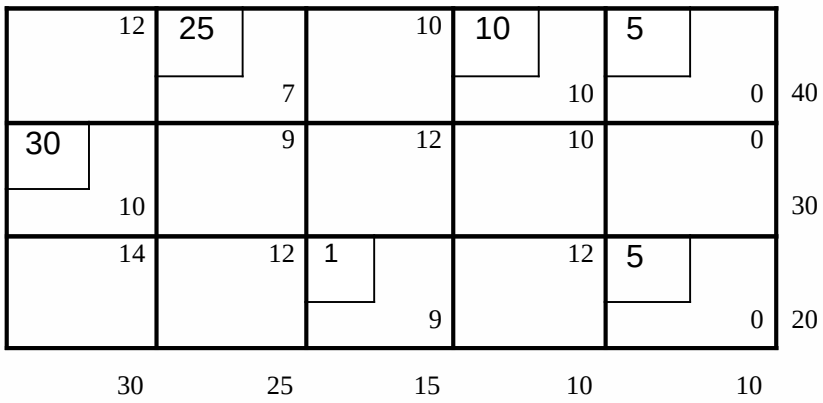
Through Radio, 30 million people must be reached in the age group 13-18.

Through Magazines, 15 million people must be reached in the age group 26-35.

And the total minimum cost of attaining the minimum exposure level is Rs. 71 lakhs.

Note : since one of Δ_{ij} in the second solution is zero. This solution is not unique, alternate solution also exists.

ii) The required solution is given by :



The total cost for this allotment is

$25 \times 7 = 175$
$10 \times 10 = 100$
$15 \times 9 = 135$
$30 \times 10 = \underline{300}$
$= \underline{710}$

i.e. Rs. 71 Lakhs.

Example 6.10

A company wishes to determine an investment strategy for each of the next four years. Five investment types have been selected, investment capital has been allocated for each of the coming four years, and maximum investment levels have been established for each investment type. An assumption is that amounts invested in any year will remain invested until the end of the planning horizon of four years. The following table summarises the data for this problem. The values in the body of the table represent net return on investment of one rupee upto the end of the planning horizon. For example, a rupee invested in investment type B at the beginning of year 1 will grow to Rs. 1.90 by the end of the fourth years, yielding a net return of Re. 0.90.

Investment made at the beginning of year	*Investment type*					*Rupees available (in 000's)*
	A	*B*	*C*	*D*	*E*	
	NET RETURN DATA					
1	*0.80*	*0.90*	*0.60*	*0.75*	*1.00*	*500*
2	*0.55*	*0.65*	*0.40*	*0.60*	*0.50*	*600*
3	*0.30*	*0.25*	*0.30*	*0.50*	*0.20*	*750*
4	*0.15*	*0.12*	*0.25*	*0.35*	*0.10*	*800*
Maximum Rupees Investment (in 000's)	*750*	*600*	*500*	*800*	*1,000*	

The objective in this problem is to determine the amount to be invested at the beginning of each year in an investment type so as to maximize the net rupee return for the four-year period.

Solve the above transportation problem and get an optimal solution. also calculate the net return on investment for the planning horizon of four-year period.

Solution

We note that this transportation problem is an unbalanced one and it is a MAXIMISATION problem. As a first step, we will balance this transportation problem.

Step 1:

Investment type

Years	A	B	C	D	E	Available Rupees (in 000's)
1	Net Return Data (Re.)					
	0.80	*0.90*	*0.60*	*0.75*	*1.00*	500
2	*0.55*	*0.65*	*0.40*	*0.60*	*0.50*	600
3	*0.30*	*0.25*	*0.30*	*0.50*	*0.20*	750
4	*0.15*	*0.12*	*0.25*	*0.35*	*0.10*	800
Dummy	0	0	0	0	0	1,000
Maximum rupees investment in ('000)	750	600	500	800	1,000	3,650

Step 2:

We shall now convert the above transportation problem (a profit matrix) into a loss matrix by subtracting all the elements from the highest value in the table viz. Re. 1.00

Investment type

Years	A	B	C	D	E	Available Rupees (in 000's)
1	Net Loss Data (Re.)					
	0.20	*0.10*	*0.40*	*0.25*	*0*	500
2	*0.45*	*0.35*	*0.60*	*0.40*	*0.50*	600
3	*0.70*	*0.75*	*0.70*	*0.50*	*0.80*	750
4	*0.85*	*0.88*	*0.75*	*0.65*	*0.90*	800
Dummy	1.00	1.00	1.00	1.00	1.00	1,000
Maximum rupees investment in ('000)	750	600	500	800	1,000	3,650

For convenience, let us express the net loss data in the body of the above table in paise. Thereafter, we shall apply VAM to get an initial solution.

Years	A	B	C	D	E	Rs. available (000's)	Difference
1	20	10	40	25	500 0	500/0	10 - - - -
2	45	600 35	60	40	50	600/0	5 5 - - -
3	70	75	70	750 50	80	750/0	20 20 20 - -
4	250 85	88	500 75	50 65	90	800/750/ 250/0	10 10 10 10 10
Dummy	500 100	100	100	100	500 100	100/500/0	0 0 0 0 0
Maximum rupee investment in (000)	750/ 500/0	600/0	500/0	800/ 50/0	1000/ 500/0		
differences	25	25	20	15	50		
	25	40	10	10	30		
	15	-	5	15	10		
	15	-	25	35	10		
	15	-	25	-	10		

The initial solution got above by VAM is given below :

Years	A	B	C	D	E
1	20	e 10	40	25	500 0
2	45	600 35	60	40	50
3	70	75	70	750 50	80
4	250 85	88	500 75	50 65	90
Dummy	500 100	100	100	100	500 100

We will now test the above solution for optimality.

The total number of allocations should be $n1 \square n \square 1 \square 9$ but there are only 8 allocations in the above solution which are one less than 9, hence the initial solution found above is degenerate. We introduce a small quantity e in an independent least cost cell which is (1, B) in this case t make the total number of allocations equal to 9. we introduce $u_i's$ and $v_j's$ such that $\lceil_{ij} \square C_{ij} \square u_i \square v_j \; for \; i, j \square 1, - - 5$ we assume that $u_4 \square 0$ and various $u_i s, v_j s$ and $\lceil_{ij}'s$ are calculated as below.

Years	A	B	C	D	E	
1	20	e - □ 10	50	45	500 + □	$u_1 \square\square 85$
2	20	600 35	45	35	25	$u_2 \square\square 60$
3	0	-5	10	750 50	10	$u_3 \square\square 15$
4	250 85	-7	500 75	50 65	5	$u_4 \square 0$
Dummy	500 100	-10 + □	10	20	500 - □ 100	$u_5 \square 15$
	$u_1 \square 85$	$u_2 \square 95$	$u_3 \square 75$	$u_4 \square 65$	$u_5 \square 85$	

Since some of the Δ_{ij}'s are negative, the above initial solution is not optimal. Introducing in the cell (Dummy, B) with most negative Δ_{ij} an assignment . The value of and the new solution as obtained from above is shown below. The values of $u_i s, v_j s$ are also calculated. The solution satisfies the conditions of optimality. The condition $\Delta_{ij} = C_{ij} - u_i - v_j \geq 0$ for non allocated cells is also fulfilled.

Investment Type

Years	A	B	C	D	E	
1	20	10	50	45	500 0	$u_1 = 85$
2	10	600 35	35	25	10	$u_2 = 50$
3	0	575	10	750 50	100	$u_3 = 15$
4	250 85	3	500 75	50 65	5	$u_4 = 0$
Dummy	500 100	e 100	10	20	500 100	$u_5 = 15$
	$u_1 = 85$	$u_2 = 85$	$u_3 = 75$	$u_4 = 65$	$u_5 = 85$	

Since all Δ_{ij} are positive, hence, the second solution obtained above is optimal. The allocation is given below :

In the year	Investment type	Amount (in 000's)
1	E	500
2	B	600
3	D	750
4	A	250
4	C	500
4	D	50

The net return on investment for the planning horizon of four years period is given by :

500 × 1.0 + 600 × 0.65 + 750 × 0.50 + 250 × 0.15 + 500 × 0.25 + 50 × 0.35 = Rs. 1,445 thousands.

Example 6.11

A leading firm has three auditors. Each auditor can work upto 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours, project 2 will take 140 hours, the project 3 will take 160 hours. The amount per hour that can be billed for assigning each auditor to each project is given in Table 1 :

Auditor	*Project*		
	1	*2*	*3*
	Rs.	*Rs.*	*Rs.*
1	*1,200*	*1,500*	*1,900*
2	*1,400*	*1,300*	*1,200*
3	*1,600*	*1,400*	*1,500*

Formulate this as a transportation problem and find the optimal solution. Also find out the maximum total billings during the next month.

Solution

The given information can be tabulated in following transportation

Example:

	Project			
Auditor	*1*	*2*	*3*	*Time*
	Rs.	*Rs.*	*Rs.*	*available (hours)*
1	*1,200*	*1,500*	*1,900*	*160*
2	*1,400*	*1,300*	*1,200*	*160*
3	*1,600*	*1,400*	*1,500*	*160*
Time required (hrs)	*130*	*140*	*160*	

The given problem is an unbalanced transportation. Introducing a dummy project to balance it, we get.

Auditor	*1*	*2*	*3*	*Dummy*	*Time*
	Rs.	*Rs.*	*Rs.*	*(Rs.)*	*available (hours)*
1	*1,200*	*1,500*	*1,900*	*0*	*160*
2	*1,400*	*1,300*	*1,200*	*0*	*160*
3	*1,600*	*1,400*	*1,500*	*0*	*160*
Time required (hrs)	*130*	*140*	*160*	*50*	*480*

The objective here is to maximize total billing amount of the auditors. For achieving this objective, let us convert this maximisation problem into a minimization problem by subtracting all the elements of the above payoff matrix from the highest payoff i.e. Rs. 1,900.

Auditor	*1 Rs.*	*2 Rs.*	*3 Rs.*	*Dummy (Rs.)*	*Time available (hours)*
1	*700*	*400*	*0*	*1,900*	*160*
2	*500*	*600*	*700*	*1,900*	*160*
3	*300*	*500*	*400*	*1,900*	*160*
Time required (hrs)	*130*	*140*	*160*	*50*	*480*

Now, let us apply vogel's Approximation Method to the above matrix for finding the initial feasible solution.

Project (Figure of payoff's in Rs. 00's)

Auditor	*1*	*2*	*3*	*Dummy*	*Time available*	*Difference*
1	*7*	*4*	160 *0*	*19*	*160/0*	*4/-/-/-*
2	*5*	110 *6*	*7*	50 *19*	*160/50/0*	*1/1/13/13*
3	13 *3*	3 *5*	*4*	*19*	*160/30/0*	*1/2/14/-*
Time Required	*130/0*	*140/110/0*	*160/0*	*50/0*		
Difference	*2/2/-/-*	*1/1/1/*	*4/-/-*	*0/0/0*		

The initial solution is given below. It can be seen that it is a degenerate solution since the number of allocations is 5. In order to apply optimality test, the total number of allocations should be *6 (= m + n – 1)*. To make the initial solution a non-degenerate, we introduce a very small quantity in the least cost independent cell which is cell of Auditor 3, Project 3.

Auditor	*1*	*2*	*3*	*Dummy*	*Time available*
1	*7*	*4*	160 *0*	*19*	*160*
2	*5*	110 *6*	*7*	50 *19*	*160*
3	13 *3*	3 *5*	e *4*	*19*	*160*
Time Required	*130*	*140*	*160*	*50*	

Introduce u_i's and v_j's such that $\lfloor_{ij} \lfloor C_{ij} \lfloor u_i \lfloor v_j$ $for\ i, \lfloor 1\ to\ 3;\ j \lfloor 1, 2, 3, dummy$. To determine the values of u_i's and v_j's, we assume $u_i \lfloor 0$, values of other variables i.e. u_i's, $\lfloor_{ij}$s are calculated as follows :

6.8 LET US SUM UP

Thus The Transportation Problem is a particular case of the Linear Programming Problem. There are 3 methods of obtaining the initial basic feasible solution to the transportation problem.

1) The North West Corner Rule (NWCR)
2) The Least Cost Method / The Matrix Minima Method (MMM)
3) Vogel's Approximation Method (VAM)

The initial basic feasible solution obtained by VAM is very close to the optimum solution. Further the MODI method is applied to obtain the optimum solution of the Transportation Problem.

The variations in the Transportation Problem are :

1) The Maximisation Case : Before obtaining the initial basic feasible solution to the Transportation problem, it is necessary that the objective function is to minimize. The maximisation Problem can be converted to the Minimisation Problem by any one of the following method.

i) Multiply all the elements of maximisation variable by - 1.
ii) Subtract all the elements of maximisation variable from the highest element.

2) The Unbalanced Problem : The Transportation Problem is a balanced problem when the row total = column total. When it is an unbalanced Transportation Problem it is necessary to introduce a dummy row or a dummy column according to the requirement of the problem.

3) Restrictive Allocation.
4) Multiple Optimum Solutions.

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6.10 UNIT END EXERCISE

1. A steel company has three open hearth furnaces and five rolling mills. Transportation costs (rupees per quintal) for shipping steel from furnaces to rolling mills are shown in the following table :

	M_1	M_2	M_3	M_4	M_5	Supply
F_1	4	2	3	2	6	8
F_2	5	4	5	2	1	12
F_3	6	5	4	7	7	14
Demand	4	4	6	8	8	

What is the optimal shipping schedule?

2. Consider the following unbalanced transportation problem.

To

		I	II	III	Supply
From	A	5	1	7	10
	B	6	4	6	80
	C	3	2	5	15
	Demand	75	20	50	

Since there is not enough supply; some of the demands at these destinations may be satisfied. Suppose there are penalty costs for every unsatisfied demand unit which are given by 5, 3 and 2 for destinations I, II and III, respectively. Find the optimal solution.

3. A company produces a small component for an industrial product and distributes it to five wholesalers at a fixed delivered price of Rs. 2.50 per unit. Sales forecasts indicate that monthly deliveries will be 3,000, 3,000, 10,000, 5,000 and 4,000 units to wholesalers I, II, III, IV and V respectively. The monthly production capacities are 5,000, 10,000 and 12,500 at plants W, X and Y, respectively. Respective direct costs of production of each unit are Re 1.00, Re 0.90, and Re 0.80 at plants W, X and Z. Transportation costs of shipping a unit from a plant to a wholesaler are as follows.

Wholesaler

		I	II	III	IV	V
Plant	W	0.05	0.07	0.10	0.25	0.15
	X	0.08	0.06	0.09	0.12	0.14
	Y	0.10	0.09	0.08	0.10	0.15

Find how many components each plant supplies to each wholesaler in order to maximize its profit.

4. Obtain an optimum basic feasible solution to the following degenerate transportation problem.

		To			
		A	B	C	Supply
From	X	7	3	4	2
	Y	2	1	3	3
	Z	3	4	6	5
	Demand	4	1	5	

5. A manufacturer wants to ship 8 loads of his product shown in the table. The matrix gives the mileage from origin to destination. Shipping costs are Rs. 10 per load per mile. What shipping schedule should be used?

	D_1	D_2	D_3	Supply
O_1	50	30	220	1
O_2	90	45	170	3
O_3	250	200	50	4
Demand	4	2	2	

6. A cement company has three factories which manufacture cement which is then transported to four distribution centres. The quantity of monthly production of each factory, the demand of each distribution centre and the associated transportation cost per quintal are given below :

Factory	Distribution centres				Monthly production (in quantals)
	W	X	Y	Z	
A	10	8	5	4	7,000
B	7	9	15	8	8,000
C	6	10	14	8	10,000
Monthly demand (in quintals)	6,000	6,000	8,000	5,000	25,000

i) Suggest the optimum transportation schedule.

ii) Is there any other transportation schedule which is equally attractive? If so, write that.

iii) If the company wants that at least 5,000 quintals of cement are transported from factory C to distribution centre Y, will the transportation schedule be any different? If so, what will be the new optimum schedule and the effect on cost?

7. A company has three plants of cement which has to be transported to four distribution centres. With identical costs of production at the three plants, the only variable costs involved are transportation costs. The monthly demand at the four distribution centres and the distance from the plants to the distribution centres (in kms) are given below :

Factory	Distribution centres				Monthly production (tonnes)
	W	X	Y	Z	
A	500	1,000	150	800	10,000
B	200	700	500	100	12,000
C	600	400	100	900	8,000
Monthly demand (tonnes)	9,000	9,000	10,000	4,000	

The transport charges are Rs. 10 per tonne per kilometer. Suggest optimum transportation schedule and indicate the total minimum transportation cost. If, for certain reasons, route from plant C to distribution centre X is closed down, will the transportation scheme change? If so, suggest the new schedule and effect on total cost.

8. A company has three factories A, B and C and four distribution centres W, X, Y and Z. With identical costs of production at the three factories, the only variable costs involved are transportation costs. The monthly production at the three factories is 5,000 tonnes, 6,000 tonnes and 2,500 tonne respectively. The monthly demand at the four distribution centres is 6,000 tonnes, 4,000 tonnes, 2,000 tonnes and 15,000 tonnes, respectively. The transportation cost per tonne from different factories to different are given below :

Factory	Distribution centres			
	W	X	Y	Z
A	3	2	7	6
B	7	5	2	3
C	2	5	4	5

i) Suggest the optimum transportation schedule. What will be the minimum transportation cost?

ii) If the transportation cost from factory C to centre Y increases by Rs. 2 per tone, will the optimum transportation schedule change? Why?

iii) If the company wants to transport at least 1,000 tonnes of the product from factory A to centre Z, will the solution in (i) above change? If so, what will be the new schedule and the effect on total transportation cost?

9. A company has three plants and four warehouses. The supply and demand in units and the corresponding transportation costs are given. The table below has been taken from the solution procedure of the transportation problem :

Plant	Distribution Centres				Supply
	I	II	III	IV	
A	5	10	10 4	5	10
B	20 6	8	7	5 2	25
C	5 4	10 2	5 5	7	20
Demand	25	10	15	5	55

Answer the following questions, giving brief reasons :

1. Is this solution feasible?
2. Is this solution degenerate?
3. Is this solution optimum?
4. Does this problem have more than one optimum solution? If so, show all of them?
5. If the cost for the route B-III is reduced from Rs. 7 to Rs. 6 per unit, what will be the optimum solution?

10. A company has three plants in which it produces a standard product. It has four agencies in different parts of the country where this product is sold. The production cost varies from factory to factory and the selling price from market to market. The shipping cost per unit of the product from each plant to each of the agencies is known and is stable. The relevant data are given in the following table :

a)

Plant	Weekly production capacity (Units)	Unit production cost (in Rs.)
1	400	18
2	300	24
3	800	20

b) Shipping cost (in Rs.) per unit :

		To Agency			
		1	2	3	4
From Plant	1	2	5	7	3
	2	8	4	6	2
	3	3	4	4	5

c)

Agency	Demand (Units)	Selling price (Rs.)
1	300	32
2	400	35
3	300	31
4	500	36

Determine the optimum plan so as to maximize the profits.

11. ABC Enterprises is having three plants manufacturing dry-cells, located at different locations. Production cost differs from plant to plant. There are five sales offices of the company located in different regions of the country. The sales prices can differ from region to region. The shipping cost from plant to each sales office and other data are given by following table :

PRODUCTION DATA TABLE

Production cost per unit	Max. capacity in number of units	Plant number
20	150	1
22	200	2
18	125	3

Shipping costs :

	Sales office 1	Sales office 2	Sales office 3	Sales office 4	Sales office 5
Plant 1	1	1	4	9	4
Plant 2	9	7	8	3	3
Plant 3	4	5	3	2	7

Demand & sales price

Demand	0	100	75	45	125
Sales Price	0	32	31	34	29

Find the production and distribution schedule most profitable to the company.

12. A company has seven manufacturing units situated in different parts of the country. Due to recession it is proposing to close four of these units and to concentrate production on the three remaining units. Production in the remaining units will actually be increased from present levels and will require an increase in the personnel employed in them. Personnel at the units to be closed have signified a willingness to move to any of three remaining units and the company is willing to provide them with removal costs. The technology of production is different to some degree at each unit and retraining expenses will be incurred on transfer. Not all existing personnel can be absorbed by transfer and a number of redundancies will arise. Cost of redundancy is given is a general figure at each unit to be closed.

Number employed	A	B	C	D
Retraining cost :	200	400	300	200
	Rs. (thousand) per person			
Transfer to:	A	B	C	D
Unit E	0.5	0.4	0.6	0.3
Unit F	0.6	0.4	0.6	0.3
Unit G	0.5	0.3	0.7	0.3
Removal costs :	Rs. (thousand) per person			
Transfer to:	A	B	C	D
Unit E	2.5	3.6	3.4	3.7
Unit F	2.4	4.6	3.4	1.7
Unit G	2.5	2.7	3.3	2.7
Redundancy payments :	Rs. (thousand) per person			
	A	B	C	D
	6.0	5.0	6.0	7.0
Additional personnel required at units remaining open :		E	F	G
		350	450	200

Use the transportation method to obtain an optimum solution to the problem of the cheapest (least cost) means to transfer personnel from the units to be closed to those will be expanded.

7

ASSIGNMENT PROBLEM

Unit Structure :

1. Objectives
2. Introduction
3. Mathematical Model of Assignment Problem
4. Solution methods of assignment problem
   - 7.3.1 Enumeration method
   - 7.3.2 Simplex method
   - 7.3.3 Transportation method
   - 7.3.4 Hungarian method
4. Check your progress
5. Additional Problems
6. Let us sum up
7. Future References
8. Unit End Exercise

1. OBJECTIVES

After going through this chapter you will be able to :

- Understand what is an assignment problem.
- Distinguish between the transportation problem and an assignment problem.
- Know the algorithm to solve an assignment problem.
- Formulate an assignment problem.
- Identify different types of assignment problems.
- Solve an assignment problem.

2. INTRODUCTION

An assignment problem is a particular case of transportation problem where the objective is to assign a number of resources to an equal number of activities so as to minimize total cost or maximize total profit of allocation.

The problem of assignment arises because available resources such as men, machines, etc., have varying degrees of efficiency for performing different activities. Therefore, cost, profit or time of performing the different activities is different. Thus, the problem is : How should the assignments be made so as to optimize the given objective.

Some of the problems where the assignment technique may be useful are : Assignment of workers to machines, salesmen to different sales areas, clerks to various checkout counters, classes to rooms, vehicles to routes, contracts to bidders, etc.

7.2 MATHEMATICAL MODEL OF AN ASSIGNMENT PROBLEM

Given n resources (or facilities) and n activities (or jobs), and effectiveness (in terms of cost, profit, time, etc.) of each resource (facility) for each activity (job), the problem lies in assigning each resource to one and only one activity (job) so that the given measure of effectiveness is optimized. The data matrix for this problem is shown in Table 7.1.

From Table 7.1, it may be noted that the data matrix is the same as the transportation cost matrix except that supply (or availability) of each of the resources and the demand at each of the destinations is taken to be one. It is due to this fact that assignments are made on a one-to-one basis.

Table 7.1 Data Matrix

Resources (workers)	J_1	Activities (jobs) J_2	...	J_n	supply
W_1	c_{11}	c_{12}	...	c_{1n}	1
W_2	c_{21}	c_{22}	...	c_{2n}	1
.	.	.		.	.
.	.	.		.	.
.	.	.		.	.
W_n	c_{n1}	c_{n2}	...	c_{nn}	1
Demand	1	1	...	1	n

Let x_{ij} denote the assignment of facility i to job j such that

$$x_{ij} = \begin{cases} 1 \text{ if facility i is assigned to job j} \\ 0 \text{ otherwise} \end{cases}$$

Then, the mathematical model of the assignment problem can be stated as:

Minimize $Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$ subject to the constraints.

$\sum_{j=1}^{n} x_{ij} = 1$ for all i (resource availability)

$\sum_{i=1}^{n} x_{ij} = 1$, for all j (activity requirement) and $x_{ij} = 0$ or 1, for all i and j where c_{ij} represents the cost of assignment of resource i to activity j.

From the above discussion, it is clear that the assignment problem is a variation of the transportation problem with two characteristics : (i) the cost matrix is a square matrix, and (ii) the optimal solution for the problem would always be such that there would be only one assignment in a given row or column of the cost matrix.

Remark : In an assignment problem if a constant is added to or subtracted from every element of any row or column in the given cost matrix, an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other matrix.

3. SOLUTION METHODS OF ASSIGNMENT PROBLEM

An assignment problem can be solved by the following four methods :

- Enumeration method
- Simplex method
- Transportation method
- Hungarian method

7.3.1 Enumeration Method :

In this method, a list of all possible assignments among the given resources (men, machines, etc.) and activities (jobs, sales areas, etc.) is prepared. Then an assignment involving the minimum cost (or maximum profit), time or distance is selected. If two or more assignments have the same minimum cost (or maximum profit), time or distance, the problem has multiple optimal solutions.

In general, if an assignment problem involves n workers / jobs, then there are in total n! possible assignments. For example, for an n = 5 workers/ jobs problem, we have to evaluate a total of 5! or 120

assignments. However, when n is large, the method is unsuitable for manual calculations. Hence, this method is suitable only for small n.

2. Simplex Method :

Since each assignment problem can be formulated as a 0 or 1 integer linear programming problem, such a problem can also be solved by the simplex method. As can be seen in the general mathematical formulation of the assignment problem, there are n × n decision variables and n + n or 2n equalities. In particular, for a problem involving 5 workers / jobs, there will be 25 decision variables and 10 equalities. It is, again, difficult to solve manually.

3. Transportation Method :

Since an assignment problem is a special case of the transportation problem, it can also be solved by transportation methods discussed in Chapter 6. However, every basic feasible solution of a general assignment problem having a square payoff matrix of order n should have n + n - 1 = 2n - 1 assignments. But due to the special structure of this problem, any solution cannot have more than n assignments. Thus, the assignment problem is inherently degenerate. In order to remove degeneracy, (n-1) number of dummy allocations (deltas or epsilons) will be required in order to proceed with the algorithm solving a transportation problem. Thus, the problem of degeneracy at each solution makes the transportation method computationally inefficient for solving an assignment problem.

4. Hungarian Assignment Method (HAM) :

It may be observed that none of the three working methods discussed earlier to solve an assignment is efficient. A method, designed specially to handle the assignment problems in an efficient way, called the Hungarian Assignment Method, is available, which is based on the concept of opportunity cost. It is shown in Table 7.1 and is discussed here. For a typical balanced assignment problem involving a certain number of persons and an equal number of jobs, and with an objective function of the minimization type, the method is applied as listed in the following steps :

Step 1 Locate the smallest cost element in each row of the cost table. Now subtract this smallest element from each element in that row. As a result, there shall be at least one zero in each row of this new table, called the Reduced Cost Table.

Step 2 In the Reduced Cost Table obtained, consider each column and locate the smallest element in it. Subtract the smallest value from every other entry in the column. As a consequence of this action, there would be at least one zero in each of the rows and columns of the second reduced cost table.

Step 3 Draw the minimum number of horizontal and vertical lines (not the diagonal ones) that are required to cover all the ‘zero’ elements. If the number of lines drawn is equal to n (the number of rows/columns) the solution is optimal and proceed to step 6. If the number of lines drawn is smaller than n, go to step 4.

Step 4 Select the smallest uncovered (by the lines) cost element. Subtract this element from all uncovered elements including itself and add this element to each value located at the intersection of any two lines. The cost elements through which only one line passes remain unaltered.

Step 5 Repeat steps 3 and 4 until an optimal solution is obtained.

Step 6 Given the optimal solution, make the job assignments as indicated by the ‘zero’ elements. This is done as follows :

a)Locate a row which contains only one ‘zero’ element. Assign the job corresponding to this element to its corresponding person. Cross out the zero’s, if any, in the column corresponding to the element, which is indicative of the fact that the particular job and person are no more available.

b)Repeat (a) for each of such rows which contain only one zero. Similarly, perform the same operation in respect of each column containing only one ‘zero’ element, crossing out the zero(s), if any, in the row in which the element lies.

c)If there is row or column with only a single ‘zero’ element left, then select a row/column arbitrarily and choose one of the jobs (or persons) and make the assignment. Now cross the remaining zeros in the column and row in respect of which the assignment is made.

d) Repeat steps (a) through (c) until all assignment are made.

e) Determine the total cost with reference to the original cost table.

Example 7.1

Solve the assignment optimal solution using HAM.

Table 7.1 Time Taken (in minutes) by 4 workers

Worker	A	B	Job C	D
1	45	40	51	67
2	57	42	63	55
3	49	52	48	64
4	41	45	60	55

The solution to this problem is given here in a step-wise manner.

Step 1 The minimum value of each row is subtracted from all elements in the row. It is shown in the reduced cost table, also called Opportunity Cost Table, given in Table 7.2.

Table 7.2 Reduced Cost Table 1

			Job		
	A	B		C	D
Worker					
1	5	0		11	27
2	15	0		21	13
3	1	4		0	16
4	0	4		19	14

Step 2 For each column of this table, the minimum value is subtracted from all the other values.

Obviously, the columns that contain a zero would remain unaffected by this operation. Here only the fourth column values would change. Table 7.3 shows this.

Table 7.3 Reduced Cost Table 2

		Job		
Worker	A	B	C	D
1	5	0	11	14
2	15	0	21	0
3	1	4	0	3
4	0	4	19	1

Step 3 Draw the minimum number of lines covering all zeros. As a general rule, we should first cover those rows/columns which contain larger number of zeros. Table 7.3 is reproduced in Table 7.4 and the lines are drawn.

Table 7.4 Reduced Cost Table 3

		Job		
Worker	A	B	C	D
1	5	0	11	14
2	15	0	21	0
3	1	4	0	3
4	0	4	19	1

Step 4 Since the number of lines drawn is equal to 4 (=n), the optimal solution is obtained. The assignments are made after scanning the rows and columns for unit zeros. Assignments made are shown with squares, as shown in Table 7.5

Table 7.5 Assignment of Jobs

Worker	Job A	Job B	Job C	Job D
1	5	[0]	11	14
2	15	~~0~~	21	[0]
3	1	4	[0]	3
4	[0]	4	19	1

Assignments are made in the following order. Rows 1, 3 and 4 contain only one zero each. So assign 1-B, 3-C and 4-A. Since worker 1 has been assigned job B, we cross the zero in the second column of the second row. After making these assignments, only worker 2 and job D are left for assignment. The final pattern of assignments is 1-B, 2-D, 3-C and 4-A, involving a total time of 40 + 55 + 48 + 41 = 184 minutes. This is the optimal solution to the problem - the same as obtained by enumeration and transportation methods.

Example 7.2

Using the following cost matrix, determine (a) optimal job assignment, and (b) the cost of assignments.

Machinist	Job 1	2	3	4	5
A	10	3	3	2	8
B	9	7	8	2	7
C	7	5	6	2	4
D	3	5	8	2	4
E	9	10	9	6	10

Iteration 1 Obtain row reductions.

Table 7.6 Reduced Cost Table 1

Machinist	Job 1	2	3	4	5
A	8	1	1	0	6
B	7	5	6	0	5
C	5	3	4	0	2
D	1	3	6	0	2
E	3	4	3	0	4

Iteration 2 Obtain column reductions and draw the minimum number of lines to cover all zeros.