**Table 7.7 Reduced Cost Table 2**

| Machinist | Job | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| A | 7 | 0 | 0 | 0 | 4 |
| B | 6 | 4 | 5 | 0 | 3 |
| C | 4 | 2 | 3 | 0 | 0 |
| D | 0 | 2 | 5 | 0 | 0 |
| E | 2 | 3 | 2 | 0 | 2 |

Since the number of lines covering all zeros is less than the number of columns/ rows, we modify the Table 7.7. The least of the uncovered cell values is 2. This value would be subtracted from each of the uncovered values and added to each value lying at the intersection of lines (corresponding to cells A-4, D-4, A-5 and D-5). Accordingly, the new table would appear as shown in Table 7.8.

**Iteration 3**

**Table 7.8 Reduced Cost Table 3**

| Machinist | Job | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| A | 7 | $\boxed{0}$ | 0 | 2 | 6 |
| B | 4 | 2 | 3 | $\boxed{0}$ | 3 |
| C | 2 | 0 | 1 | 0 | $\boxed{0}$ |
| D | $\boxed{0}$ | 2 | 5 | 2 | 2 |
| E | 0 | 1 | $\boxed{0}$ | 0 | 2 |

The optimal assignments can be made as the least number of lines covering all zeros in Table 7.8 equals 5. Considering rows and columns, the assignments can be made in the following order :

i) Select the second row, Assign machinist B to job 4. Cross out zeros at cells C-4 and E-4.

ii) Consider row 4. Assign machinist D to job 1. Cancel the zero at cell E-1.

iii) Since there is a single zero in the fifth row, put machinist E to job 3 and cross out the zero at A-3.

iv) There being only a single zero left in each of the first and third rows, we assign job 2 to machinist A and job 5 to C.

The total cost associated with the optimal machinist job assignment pattern A-2, B-4, C-5, D-1 and E-3 is 3 + 2 + 4 + 3 9 = 21.

## 7.4 CHECK YOUR PROGRESS

1. What is an assignment problem? Give two applications.
2. Give the mathematical formulation of an assignment problem. How does it differ from a transportation problem?
3. Explain the conceptual justification that an assignment problem can be viewed as a linear programming problem.
4. Explain the difference between a transportation problem and an assignment problem.
5. State and discuss the methods for solving an assignment problem. How is the Hungarian method better than other methods for solving an assignment problem?
6. Solve the following assignment problem by (a) enumeration method, and (b) Hungarian assignment method.

Time (in minutes)

| Worker | Job 1 | Job 2 | Job 3 |
|---|---|---|---|
| A | 4 | 2 | 7 |
| B | 8 | 5 | 3 |
| C | 4 | 5 | 6 |

7. ABC Company is engaged in manufacturing 5 brands of packed snacks. It has five manufacturing setups, each capable of manufacturing any of its brands one at a time. The costs to make a brand on these setups vary according to the following table.

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ |
|---|---|---|---|---|---|
| $B_1$ | 4 | 6 | 7 | 5 | 11 |
| $B_2$ | 7 | 3 | 6 | 9 | 5 |
| $B_3$ | 8 | 5 | 4 | 6 | 9 |
| $B_4$ | 9 | 12 | 7 | 11 | 10 |
| $B_5$ | 7 | 5 | 9 | 8 | 11 |

Assuming five setups are $S_1$, $S_2$, $S_3$, $S_4$ and $S_5$ and five brands are $B_1$, $B_2$, $B_3$, $B_4$ and $B_5$. Find the optimum assignment of products on these setups resulting in the minimum cost.

8. Five employees of a company are to be assigned to five jobs which can be done by any of them. Because of different number of years with the firm, the workers get different wages per hour. These are : Rs. 15 per hour for A, B and C each, and Rs. 13 per hour for D and E each. The amount of time taken (in hours) by each employee to do a given job is given in the following table. Determine the assignment pattern that (a)

minimizes the total time taken, and (b) minimizes the total cost of getting five units of work done.

| Job | Employee | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | E |
| 1 | 7 | 9 | 3 | 3 | 2 |
| 2 | 6 | 1 | 6 | 6 | 5 |
| 3 | 3 | 4 | 9 | 10 | 7 |
| 4 | 1 | 5 | 2 | 2 | 4 |
| 5 | 6 | 6 | 9 | 4 | 2 |

9. A departmental head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimates of the times that each man would take to perform each task is given below in the matrix :

| | | Tasks | | | |
|---|---|---|---|---|---|
| | | I | II | III | IV |
| | A | 8 | 26 | 17 | 11 |
| | B | 13 | 28 | 4 | 26 |
| Subordinates | C | 38 | 19 | 18 | 15 |
| | D | 19 | 26 | 24 | 10 |

How should the tasks be allocated to subordinates so as to minimize the total man-hours?

10. An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of man-hours that would be required for each job-man combination. This is given in matrix form in the following table :

| | | A | **jobs** B | C | D |
|---|---|---|---|---|---|
| | 1 | 5 | 3 | 2 | 8 |
| Men | 2 | 7 | 9 | 2 | 6 |
| | 3 | 6 | 4 | 5 | 7 |
| | 4 | 5 | 7 | 7 | 8 |

Find the optimal assignment that will result in minimum man-hours needed.

## 7.5 ADDITIONAL PROBLEMS

**Example 7.3**

Solve the following unbalanced assignment problem of minimizing total time for doing all the jobs :

| Operator | Job | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| 1 | 6 | 2 | 5 | 2 | 6 |
| 2 | 2 | 5 | 8 | 7 | 7 |
| 3 | 7 | 8 | 6 | 9 | 8 |
| 4 | 6 | 2 | 3 | 4 | 5 |
| 5 | 9 | 3 | 8 | 9 | 7 |
| 6 | 4 | 7 | 4 | 6 | 8 |

**Solution**

Since the number of operators is unequal to the number of jobs, a dummy job is created. The time consumed by any operator for the dummy job is 0.

| Operator | Job | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 6 | 2 | 5 | 2 | 6 | 0 |
| 2 | 2 | 5 | 8 | 7 | 7 | 0 |
| 3 | 7 | 8 | 6 | 9 | 8 | 0 |
| 4 | 6 | 2 | 3 | 4 | 5 | 0 |
| 5 | 9 | 3 | 8 | 9 | 7 | 0 |
| 6 | 4 | 7 | 4 | 6 | 8 | 0 |

Subtracting the smallest value in each column from all the values of that column drawing minimum number of lines to cover all zeros we have :

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 0 | 1 | 0 |
| 2 | 0 | 3 | 5 | 5 | 2 | 0 |
| 3 | 5 | 6 | 3 | 7 | 3 | 0 |
| 4 | 4 | 0 | 0 | 2 | 0 | 0 |
| 5 | 7 | 1 | 5 | 7 | 2 | 0 |
| 6 | 2 | 5 | 1 | 4 | 3 | 0 |

We find that no more than four lines are needed to cover all zeros, which is not equal to number of assignments. We note that all rows contain at least one zero, and 1 is the smallest value not covered by a line. Subtracting 1 from every uncovered value and adding 1 to every value at the intersection of two lines, we find

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 0 | 1 | 1 |
| 2 | 0 | 3 | 5 | 5 | 2 | 1 |
| 3 | 4 | 5 | 2 | 6 | 2 | 0 |
| 4 | 4 | 0 | 0 | 2 | 0 | 1 |
| 5 | 6 | 0 | 4 | 6 | 1 | 0 |
| 6 | 1 | 4 | 0 | 3 | 2 | 0 |

Drawing minimum number of lines to cover all zeros, we find

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | 0 | 1 | 1 |
| 2 | 0 | 3 | 5 | 5 | 2 | 1 |
| 3 | 4 | 5 | 2 | 6 | 2 | 0 |
| 4 | 4 | 0 | 0 | 2 | 0 | 1 |
| 5 | 6 | 0 | 4 | 6 | 1 | 0 |
| 6 | 1 | 4 | 0 | 3 | 2 | 0 |

Now the minimal number of lines is equal the number of assignments that can be made. Optimal assignment can be made by sciecting one zero in each row so that no two selected zeros are in the same column.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 2 | $\boxed{0}$ | 1 | 1 |
| 2 | $\boxed{0}$ | 3 | 5 | 5 | 2 | 1 |
| 3 | 4 | 5 | 2 | 6 | 2 | $\boxed{0}$ |
| 4 | 4 | 0 | 0 | 2 | $\boxed{0}$ | 1 |
| 5 | 6 | $\boxed{0}$ | 4 | 6 | 1 | 0 |
| 6 | 1 | 4 | $\boxed{0}$ | 3 | 2 | 0 |

Thus the optimal assignment is

Operator 1 to job 4, Operator 2 to job 1, Operator 3 to dummy 6
Operator 4 to job 5, Operator 5 to job 2, Operator 6 to job 3
Time = 2 + 2 + 0 + 5 + 3 + 4 = 16 units.

**Example 7.4**

A solicitors' firm employs typists on hourly piece-rate basis for their daily work. There are five typists for service and their charges and speeds are different. According to an earlier understanding only one job is given to one typist and the typist is paid for full hours even if he works for a fraction of an hour. Find the least cost allocation for the following.

| *Typist* | *Rate per hour (Rs.)* | *No. of pages Typed / hour* | *Job* | *No. of pages* |
|---|---|---|---|---|
| *A* | *5* | *12* | *P* | *199* |
| *B* | *6* | *14* | *Q* | *175* |
| *C* | *3* | *8* | *R* | *145* |
| *D* | *4* | *10* | *S* | *298* |
| *E* | *4* | *11* | *T* | *178* |

**Solution**

The following matrix gives the cost incurred if the ith typist (i = A,B,C,D,E) executes the *j*$^{th}$ job (j = P,Q,R,S,T)

| Typist | P | Q | R | S | T |
|---|---|---|---|---|---|
| | | | **Job** | | |
| A | 85 | 75 | 65 | 125 | 75 |
| B | 90 | 78 | 66 | 132 | 78 |
| C | 75 | 66 | 57 | 114 | 69 |
| D | 80 | 72 | 60 | 120 | 72 |
| E | 76 | 64 | 56 | 112 | 68 |

Subtracting the minimum element of each row from all its elements in turn matrix reduces to

| Typist | P | Q | R | S | T |
|---|---|---|---|---|---|
| | | | **Job** | | |
| A | 20 | 10 | 0 | 60 | 10 |
| B | 24 | 2 | 0 | 66 | 12 |
| C | 18 | 9 | 0 | 57 | 12 |
| D | 20 | 12 | 0 | 60 | 12 |
| E | 20 | 8 | 0 | 56 | 12 |

Now subtract the minimum element of each column from all its elements in turn, of matrix reduces to :

| Typist | P | Q | R | S | T |
|---|---|---|---|---|---|
| | | | **Job** | | |
| A | 2 | 2 | 0 | 4 | 0 |
| B | 6 | 4 | 0 | 10 | 2 |
| C | 0 | 1 | 0 | 1 | 2 |
| D | 2 | 4 | 0 | 4 | 2 |
| E | 2 | 0 | 0 | 0 | 2 |

Since there are only 4 lines (<5) to cover all zeros, optimal assignment cannot be made. The minimum uncovered element is 2.

We subtract the value 2 from all uncovered elements, add this value to all junction values and leave the other elements undisturbed. The revised matrix looks as

**Job**

| Typist | P | Q | R | S | T |
|---|---|---|---|---|---|
| A | 2 | 2 | 2 | 4 | 0 |
| B | 4 | 2 | 0 | 8 | 0 |
| C | 0 | 1 | 2 | 1 | 2 |
| D | 0 | 2 | 0 | 2 | 0 |
| E | 0 | 2 | 0 | 2 | 2 |

Since the minimum no. of lines required to cover all the zeros is only 4(<5), optimal assignment cannot be made at this stage also.

The minimum uncovered element is 1. Repeating the usual process again, we get the following matrix :

**Job**

| Typist | P | Q | R | S | T |
|---|---|---|---|---|---|
| A | 2 | 1 | 2 | 3 | 0 |
| B | 4 | 1 | 0 | 7 | 0 |
| C | 0 | 0 | 2 | 0 | 2 |
| D | 0 | 1 | 0 | 1 | 0 |
| E | 3 | 0 | 3 | 0 | 0 |

Since the minimum number of lines to cover all zero is equal to 5, this matrix will given optimal solution. The optimal assignment is made in the matrix below.

**Job**

| Typist | P | Q | R | S | T |
|---|---|---|---|---|---|
| A | 2 | 1 | 2 | 3 | $\boxed{0}$ |
| B | 4 | 1 | $\boxed{0}$ | 7 | $\cancel{0}$ |
| C | $\cancel{0}$ | $\boxed{0}$ | 2 | $\cancel{0}$ | 2 |
| D | $\boxed{0}$ | 1 | $\cancel{0}$ | 1 | $\cancel{0}$ |
| E | 3 | $\cancel{0}$ | 3 | $\boxed{0}$ | 3 |

| | | | Cost Rs. |
|---|---|---|---|
| Thus typist A is given Job | T | : | 75 |
| Thus typist B is given Job | R | : | 66 |
| Thus typist C is given Job | Q | : | 66 |
| Thus typist D is given Job | P | : | 80 |
| Thus typist E is given Job | S | : | 112 |
| | | | Rs. 399 |

**Note :** In this case the above solution is not unique. Alternate solution also exists.

**Example 7.5**

To stimulate interest and provide an atmosphere for intellectual discussion, a finance faculty in a management school decides to hold special seminars on four contemporary topics leasing, portfolio management, private mutual funds, swaps and options. Such seminars should be held once per week in the afternoons. However, scheduling these seminars (one for each topic, and not more than one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows :

| | Leasing | Portfolio Management | Private Mutual funds | Swaps and Options |
|---|---|---|---|---|
| Monday | 50 | 40 | 60 | 20 |
| Tuesday | 40 | 30 | 40 | 30 |
| Wednesday | 60 | 20 | 30 | 20 |
| Thursday | 30 | 30 | 20 | 30 |
| Friday | 10 | 20 | 10 | 30 |

Find an optimal schedule of the seminars. Also find out the total number of students who will be missing at least one seminar.

**Solution**

This is an unbalanced minimization assignment problem.
We first of all balance it by adding a dummy topic.

| | Leasing | Portfolio Management | Private Mutual funds | Swaps and Options | Dummy |
|---|---|---|---|---|---|
| Monday | 50 | 40 | 60 | 20 | 0 |
| Tuesday | 40 | 30 | 40 | 30 | 0 |
| Wednesday | 60 | 20 | 30 | 20 | 0 |
| Thursday | 30 | 30 | 20 | 30 | 0 |
| Friday | 10 | 20 | 10 | 30 | 0 |

Subtracting the minimum element of each column from all elements of that column, we get the following matrix :

| | Leasing | Portfolio Management | Private Mutual funds | Swaps and Options | Dummy |
|---|---|---|---|---|---|
| Monday | 40 | 20 | 50 | 0 | 0 |
| Tuesday | 30 | 10 | 30 | 10 | 0 |
| Wednesday | 50 | 0 | 20 | 0 | 0 |
| Thursday | 20 | 10 | 10 | 10 | 0 |
| Friday | 0 | 0 | 0 | 10 | 0 |

The minimum number of lines to cover all zeros is 4 which is less than the order of the square matrix (i.e.5), the above matrix will not give the optimal solution, Subtract the minimum uncovered element (=10) from all uncovered elements and add it to the elements lying on the intersection of two lines, we get the following matrix.

| | *Leasing* | *Portfolio Management* | *Private Mutual funds* | *Swaps and Options* | *Dummy* |
|---|---|---|---|---|---|
| *Monday* | *30* | *20* | *40* | *0* | *0* |
| *Tuesday* | *20* | *10* | *20* | *10* | *0* |
| *Wednesday* | *40* | *0* | *10* | *0* | *0* |
| *Thursday* | *10* | *10* | *0* | *10* | *0* |
| *Friday* | *0* | *10* | *0* | *20* | *10* |

Since the minimum number of lines to cover all zeros is 5 which is equal to the order to the matrix, the above matrix will give the optimal solution which is given below :

| | *Leasing* | *Portfolio Management* | *Private Mutual funds* | *Swaps and Options* | *Dummy* |
|---|---|---|---|---|---|
| *Monday* | *30* | *20* | *40* | $\boxed{0}$ | $\cancel{0}$ |
| *Tuesday* | *20* | *10* | *20* | *10* | $\boxed{0}$ |
| *Wednesday* | *40* | $\boxed{0}$ | *10* | $\cancel{0}$ | $\cancel{0}$ |
| *Thursday* | *10* | *10* | 0 | *10* | $\cancel{0}$ |
| *Friday* | $\boxed{0}$ | *10* | $\cancel{0}$ | *20* | *10* |

And the optimal schedule is

| | | | No. of students missing |
|---|---|---|---|
| Monday | : | Swaps and options | 20 |
| Tuesday | : | No seminar | 0 |
| Wednesday | : | Portfolio Management | 20 |
| Thursday | : | Pvt. mutual funds | 20 |
| Friday | : | leasing | <u>10</u> |
| | | | <u>70</u> |

Thus, the total number of students who will be missing at least one seminar = 70

**Example 7.6**

Four operators $O_1, O_2, O_3$ and $O_4$ are available to a manager who has to get four jobs $J_1, J_2, J_3$ and $J_4$ done by assigning one job to each operator. Given the times needed by different operators for different jobs in the matrix below :

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ |
|---|---|---|---|---|
| $O_1$ | 12 | 10 | 10 | 8 |
| $O_2$ | 14 | 12 | 15 | 11 |
| $O_3$ | 6 | 10 | 16 | 4 |
| $O_4$ | 8 | 10 | 9 | 7 |

*i)* *How should the manager assign the jobs so that the total time needed for all four jobs is minimum?*

*ii)* *If job $J_2$ is not to be assigned to operator $O_2$ what should be the assignment how much additional total time will be required?*

**Solution**

i) This is an assignment problem whose objective is to assign one job to one operator so that total time needed for all four jobs is minimum. To determine appropriate assign of jobs and operators, let us apply the assignment algorithm. Subtract the minimum element of each row from all elements of that row to get the following matrix :

| Job / Operators | $J_1$ | $J_2$ | $J_3$ | $J_4$ |
|---|---|---|---|---|
| $O_1$ | 4 | 2 | 2 | 0 |
| $O_2$ | 3 | 1 | 4 | 0 |
| $O_3$ | 2 | 6 | 12 | 0 |
| $O_4$ | 1 | 3 | 2 | 0 |

Now subtract the minimum element of each column from all elements of that column

| Operators / Job | $J_1$ | $J_2$ | $J_3$ | $J_4$ |
|---|---|---|---|---|
| $O_1$ | 3 | 1 | 0 | 0 |
| $O_2$ | 2 | 0 | 2 | 0 |
| $O_3$ | 1 | 5 | 10 | 0 |
| $O_4$ | 0 | 2 | 0 | 0 |

The minimum number of lines drawn to cover all zeros is equal to 4. Since the number of lines drawn viz., 4 is equal to the number of jobs or the number of operators, so we proceed for making the optimal assignment.

The optimal assignment is made as below :

| Job | $J_1$ | $J_2$ | $J_3$ |
| --- | --- | --- | --- |
| Operators | | | |