Applied Mathematics

Part-3

Check Your Progress:

(1) If $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ Evaluate div $(\bar{A} \times \bar{r})$

(2) Prove that

$$\text{div}\left(\frac{\log r}{r}\bar{r}\right) = \frac{1}{r}(1 + 2\log r)$$

(3) For $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$) show that the vector div $\left(\frac{\bar{r}}{r^3}\right)$ is both solenoidal and irrotational.

(4) Prove that div $(\bar{a}.\bar{r})\,\bar{a} = |\bar{a}|^2$

(5) For $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that $\nabla.(\nabla r^n) = n\,(n+1)r^{n-2}$

(6) show that the vector $\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ solenoidal.

(7) If $\bar{A} = (ax + 3y + 4z)\,\hat{i} + (x - 2y + 3z)\hat{j} + (3x + 2y - z)\hat{k}$ is solenoidal find value of a.

(7) Find the direction derivative of a scalar field $\phi = x^2\,y\,z$ at (4, -1, 2) in the direction of (3, 2, 1).

[**Hint :-** direction derivative of ϕ (x, y, z) along $\bar{a}$ is = $\bar{a}$. grad ϕ]

5.4 PROPERTIES OF GRADIENT, DIVERGENCE AND CURL

1) If $\bar{S}$ represents displacement vector, $\frac{d\bar{s}}{dt}$ represents velocity and $\frac{d^2\bar{s}}{dt^2}$ represents acceleration.

2) For $\frac{d\bar{s}}{dt}$ $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

grad $\underline{f} = \nabla\,\underline{F}$

grad $\underline{F} = \nabla.\,\underline{F}$

curl F $= \nabla \times \bar{F}$

3) grad F and curl $\bar{F}$ are vector quantities.

4) div $\bar{F}$ is scalar quantity.

5.5 LET US SUM UP

In this chapter we have learn

- Differentiation of vectors.
- Partial derivative of vectors.
- The vector differential Operator Del.(∇)
- Divergence of a vector function.
- Curl of a vector.
- Properties of divergence, gradient & curl.

5.6 UNIT END EXERCISE

1) If $A = x^2yi - 2xzj + xy^2k, \quad B = 3zi + 2yj - 2x^2k$

Find the value $\frac{\partial^2}{\partial y \partial x}(A \times B)G + (1,0,1)$

2) If $r = xi + yi + zk$ prove that $\left(\frac{1}{R}\right) = \frac{-1}{R^3} r.$

where $R = |r|$

3) Find the unit normal vector to the surface at the point(1,0,1).

4) Find the directional derivative of $f(x, y, z) = xy^2 + yz^3$ the point (1,-1,1) in the direction of (3,-1,1)

5) If $f = 3x^2y - xyj + 3y^2zk$ find div F curl F.

6) Show that the vector $f = (x + 3y)i + (y - 3z)j + (x - 2z)k$ is solenoid.

7) Show that the vector $f = (3x^2y)i + (x^3 - 2yz^2)j + (3z^2 - 2y^2z)k$ is irrotational.

8) Show that div r =3
where $r = xi + yi + zk$

9) Show that for any vector F
Div (Curl F)=0

10) If $a = a_1i + a_2i + a_3k$ and $r = xi + yj + zk$
Find Curl (r x a)

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6

DIFFERENTIAL EQUATIONS

UNIT STRUCTURE

6.1 Objective
6.2 Introduction
3. Differential Equation
4. Formation of differential equation
5. Let Us Sum Up
6. Unit End Exercise

6.1 OBJECTIVE

After going through this chapter you will able to

i. Define differential equation
ii. Order & degree of differential equation
iii. Formulate the differential equation

2. INTRODUCTION

We have already learned differential equation in XIIth . Hence we are going to discuss differential equation in brief. In this chapter we discuss only formulation of differential equation.

3. DIFFERENTIAL EQUATION

Definition:-

An equation involving independent and dependent variables and the differential coefficients or differentials is called a differential equation.

e.g. 1 $\frac{dy}{dx}=9$

x=independent variable

y= depedent variable

2 $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

3 $\frac{d^ny}{dx^n}+y=0$

These are all examples of differential equations.

The differential equation is said to be ordinary if it contains only one independent variable. All the examples of above are of ordinary differential equations.

Order and Degree of a Differential Equations:-

(i) Order:-

The order of the differential equations is the order of the highest orderal derivatives present in the function or equation.

If y = f (x) is a function, then

$\frac{dy}{dx}$ is the first order derivative,

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the second order derivative.

e.g 1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Order = 2

2) $E = Ri + L\frac{di}{dt}$

Order = 1

Degree:-

The degree of differential equation is the degree of the highest ordered derivative in the equation when it is made free from radicals and fractions.

e.g.

1 $\frac{d_2y}{dx^2} + k^2 y = 0$

order = 2, degree = 1

2 $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + Y = 0$

Order =2, degree =1

3 $y = \left(\frac{dy}{dx}\right)x + \frac{1}{\frac{dy}{dx}}$

Order=1, degree=2

4 $\sqrt[3]{\frac{dy^2}{dx^2}} = \sqrt{\frac{dy}{dx}}$

$\therefore \left(\frac{d^2y}{dx^2}\right)^{1/3} = \left(\frac{dy}{dx}\right)^{1/2}$

Cubing both sides

$$\therefore \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^{3/2}$$

Squaring both sides

$$\therefore \left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^3$$

Order=2, degree=2

Solved examples:

Example 1: Find the order and degree of the following

i) $$e = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Solution:

$$\therefore\ e\cdot\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

Squaring both sides

$$\therefore\ e^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

$\therefore$ order = 2, degree = 2

ii) $$\frac{d}{dx}\left\{x\left(\frac{d^2y}{dx^3}\right)^3\right\} + \sin(xy) = e^x$$

Solution:

$$\left(\frac{d^3y}{dx^3}\right)^3 + x\cdot 3\left(\frac{d^3y}{dx^3}\right)^2\frac{d^4y}{dx^4} + \sin(xy) = e^x$$

$\therefore$ Order = 4, degree=1

iii) $$y = x\cdot\frac{dy}{dx} + \frac{5}{\frac{dy}{dx}}$$

Solution:

$$\therefore\ y\cdot\frac{dy}{dx} = x\cdot\left(\frac{dy}{dx}\right)^2 + 5$$

$\therefore$ Order =1, degree=2

iv) $y = x \cdot \frac{dy}{dx} + 5\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Solution:

$$\therefore\ y - x \cdot \frac{dy}{dx} = 5\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Squaring on both sides

$$\left(y-x \cdot \frac{dy}{dx}\right)^2 = 25\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore\ y^2 - 2xy \cdot \frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2 = 25\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$\therefore$ Order =1, degree=2

Check your progress:

1) $\frac{\partial^2 u}{\partial x^2} + u \cdot \frac{\partial u}{\partial y}$

Ans : order =2, degree=1

2) $\left(\frac{d^3y}{dx^3}\right)^4 + 5\left(\frac{d^3y}{dx^3}\right)^7 + 7\left(\frac{d^2y}{dx^2}\right)^{11} + \frac{dy}{dx} + y = e^x$

Ans : order=3, degree=7

3) $\left(y^{11}\right)^3 + \left(y^1\right)^4 = e^x$

Ans : order=2, degree=3

4) $y^{11} + \frac{x}{y^{11}} = 1$

Ans : order=2, degree=2

5) $y^{11} = \sqrt{1+y^{12}}$

Ans : order=2, degree=2

$y^1 + x = \left(y - xy^1\right)^{-2}$

Ans : order =1, degree=3

6.4 FORMATION OF DIFFERENTIAL EQUATION

Formation of differential equation involves elimination of arbitrary consonants, in the relation of the variables.

Consider

$$y = ax^2 ------------1$$

Where y= independent variable

x = dependent variable

Differentiating equation (1) with respect to x

we have $\therefore \frac{dy}{dx} = 2ax ----------(2)$

From equation (1) we have

$$a = \frac{y}{x^2}$$

Put value of a in equation (2), we have

$$\therefore \frac{dy}{dx} = 2 \cdot \frac{y}{x^2} \cdot x$$

$$\therefore \frac{dy}{dx} = \frac{2y}{x}$$

$$\therefore x \cdot \frac{dy}{dx} = 2y$$

$$\therefore x \cdot \frac{dy}{dx} - 2y = 0$$

This is the required differential equation

Note:-

To eliminate two arbitrary constants, three equations are required. To eliminate three arbitrary constants, four equations are required.

In general to eliminate n arbitrary constants. (n+1) equations are required.

In other words elimination of n arbitrary consonants will bring us to differential equation of n^{th} order.

Solved Examples:-

Example 2: Form the differential equations if $y = c_1 \cos x + c_2 \sin x$

Solution: We have

$Y = c_1 \cos x + c_2 \sin x$ -------------(1)

This equation contains two arbitrary constants, therefore we shall require three equations to eliminate c_1 and c_2.

Differentiating equation (1) with respect to x

$$\therefore \frac{dy}{dx} = -c_1 \cos x + c_2 \cos x.$$

Again differentiate with respect to x

$$\therefore \frac{d^2y}{dx^2} = -c_1 \cos x - c_2 \sin x$$

$$\frac{d^2y}{dx^2} = -(c_1 \cos x + c_2 \sin x)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -y ---------- [\textit{from eq} ----(1)]$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0$$

This is the required differential equation.

Example 3: Form the differential equation from

x= a sin (wt+c) where a and c are arbitrary constants.

Solution: We have,

x= a sin (wt+c)--------(1)

Differentiate equation (1) with respect

$$\therefore \quad \frac{dx}{dt} = + a\cos(\cot + c) \cdot w$$

$$\therefore \quad \frac{dx}{dt} = + aw \cdot \cos(\cot + c)$$

Again differentiating w.r.t. 't'

$$\frac{d^2x}{dt^2} = -a\, w\sin(\cot + c) \cdot w$$

$$\frac{d^2x}{dt^2} = -w^2 [a \sin(\cot + c)]$$

$$\therefore \frac{d^2x}{dt^2} = -w^2 x........[u\sin g \text{ equation } 1]$$

$$\therefore \frac{d^2x}{dt^2} + w^2 x = 0$$

This is the required differential equation

Example 4: From the differential equation if y= log (ax)

Solution:

$$y = \log(ax) --------------(1)$$

Differentiate equation (1) with respect to x .

$$\therefore \quad \frac{dy}{dx} = \frac{1}{Ax} \cdot A$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \quad x \cdot \frac{dy}{dx} = 1$$

This is the required differential equation.

Example 5: Obtain the differential equation for the equation Y=cx+ c^2

Solution: we have,

$$y = cx + c^2 \text{ ------------(1)}$$

Differentiate equation (1) with repect to x

$$\therefore \frac{dy}{dx} = c$$

Put value of c in equation (1)

$$\therefore y = x \cdot \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

This is the required differential equation.

Example 6: Obtain the differential equation for the relation

$\therefore$ y= $a \cdot e^{2x} + b \cdot e^{3x}$ *Where* a,b are constants.

Solution: we have,

$$\therefore y = a \cdot e^{2x} + b \cdot e^{3x} \text{ -------------(1).}$$

Here the number of arbitrary constants is two

Hence we shall require three equations to

Eliminate and b. So we differentiate the given equations twice.

$$\therefore \frac{dy}{dx} = 2a \cdot e^{2x} + 3b \cdot e^{3x} \text{ -------------(2).}$$

$$\therefore \frac{d^2y}{dx^2} = 4a \cdot e^{2x} + 9b \cdot e^{3x} \text{ -------------(3)}$$

From equation (1) (2) & (3) elimination of a & b gives directly

$$\begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & 2 & 3 \\ \frac{d^2y}{dx^2} & 4 & 9 \end{vmatrix} = 0$$

In the determinant

1st column is LHS

Column 2nd column 2nd column contains coefficients $a \cdot e^{2x}$

Expanding the determinant

2nd column contains coefficients of $b \cdot e^{3x}$

$$y - (18 - 12) - \frac{dy}{dx}(9 - 4) + \frac{d^2y}{dx^2}(3 - 2) = 0$$

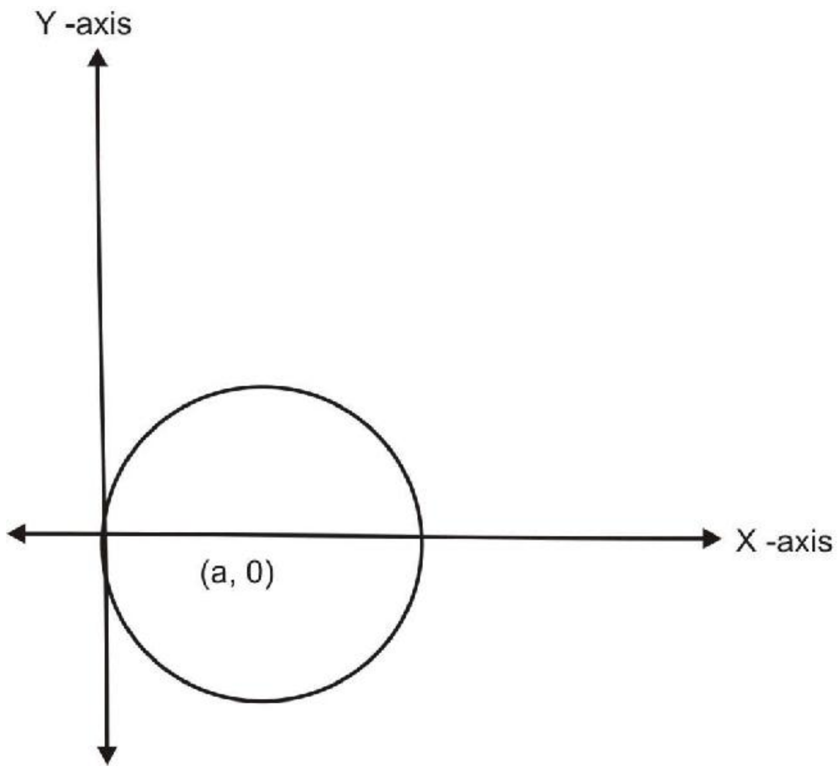
$$\therefore 6y - 5 \cdot \frac{dy}{dx} + \frac{d^2y}{dx^2} = 0$$

$$\therefore \frac{d^2y}{dx^2} - 5 \cdot \frac{dy}{dx} + 6y = 0$$

This is the required differential equation.

Example 7: Find the differential equation of all circles touching y axis at the origin and centers on x-axis

Solution:



The equation of such a circle is

$$(x-a)^2 + y^2 = a^2$$

i.e. $$x^2 - 2ax + \not{a}^2 + y^2 = \not{a}^2$$

$$\therefore x^2 + y^2 - 2ax = 0 ----------- (1)$$

Where a is the only arbitrary contents

Differentiate equation (1) with respect to x We have

$$2x + 2y \cdot \frac{dy}{dx} - 2a = 0$$

$$x^2 + y^2 - 2x \cdot \left(x + y \cdot \frac{dy}{dx} \right) = 0$$

$$x^2 + y^2 - 2x^2 - 2xy \cdot \frac{dy}{dx} = 0$$

$$-x^2 + y^2 - 2xy \cdot \frac{dy}{dx} = 0$$

$$\therefore\ 2xy\cdot\frac{dy}{dx}+x^2-y^2=0$$

Which is the require differential equation.

Check Your Progress:

1) Form the differential equation of all circles of radius a.

Ans. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)$

2) Obtain the differential equation whose general solution is given by $y=e^x\left(A\cos x+B\sin x\right)$

Ans $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$

3) Find the differential equation whose general solution is given by $y=c_1e^x+c_{2.}e^{-2x}+c_3\cdot e^{3x}$

Ans $\frac{d^3y}{dx^3}-2\cdot\frac{d^2y}{dx^2}-5\frac{dy}{dx}-6y=0$

4) Obtain the differential equations for the following:

i) $y=A\cdot e^{3x}+B\cdot e^{2x}$

Ans $\frac{d^2y}{dx^2}-5\cdot\frac{dy}{dx}+6y=0$

ii) $s=c_1e^{2t}+c_2\cdot e^{-t}$

Ans $\frac{d^2s}{dt^2}-\frac{ds}{dt}-2s=0$

iii) $y=A$ cos 2t+ B sin 2t

Ans $\frac{d^2y}{dt^2}+4y=0$

iv) $y=ax^3+bx^2$

Ans $x^2\frac{d^2y}{dx^2}-4x\cdot\frac{dy}{dx}+6y=o$

v) $x=A\cos(nt+\sim$

Ams $\frac{d^2y}{dt^2}+n^2x=0$

vi $Y=A+Bx+Cx^2$

Soln $\frac{d^2y}{dx^2}=0$

Vii $Y=\sin x+c$

Soln $\frac{dy}{dx} = \cos x$

Viii $y = (c_1 + c_2 x)e^x$

Ans $\frac{d^2y}{dx^2} - 2 \cdot \frac{dy}{dx} + y = 0$

6.5 LET US SUM UP

In this chapter we have learn

- ❖ Equation in term $\frac{dy}{dx}$ of is called differential equation.
- ❖ Degree & order of differential equation.
- ❖ Formation of differential equation while removing arbitrary constant likes A&B,&C.

6.6 UNIT END EXERCISE

1) Find the order 7 degree of Differential equation given below

i. $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{d^2y}{dx^2}\right) + 3\frac{dy}{dx} = y$

ii. $\left[i + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

iii. $\left(\frac{d^2y}{dx^2}\right)^{3/5} = \left(1 + \frac{dy}{dx}\right)$

iv. $2\frac{d^2y}{dx^2} + \sqrt[3]{1 + \left(\frac{dy}{dx}\right)^2} - y = 0$

v. $y = x\left(\frac{dy}{dx}\right) + \frac{1}{\left(dy/dx\right)}$

2) Formulate the differential equation

i. $Y = A + Blogx$

ii. $X = asin(w + +c)$

iii. $Y = c^x(Acosx + Bsinx)$

iv. $Y = e^{m\cos^{-1}x}$

v. $Y = ax^2 + bx$

vi. $Y = cx + 2c^2 + c^3$

vii. $X^2 + Y^2 = 2ax$

viii. $Y^2 = 4ax$

ix. $e^x + Ce^y = 1$

x. $Y = ACos2x + BSin2x$

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7

SOLUTION OF DIFFERENTIAL EQUATION

UNIT STRUCTURE

1. Objectives
2. Introduction
3. Solution of Differential equation
4. Solution of Differential Equation of first order and first degree
5. Let Us Sum Up
6. Unit End Exercise

7.1 OBJECTIVES

After going through this chapter you will able to

- ❖ Find general & particular solution of differential equations.
- ❖ Classification of differential equation.
- ❖ Apply particular method first find the solution of differential **equation.**

2. INTRODUCTION

We have already formed differential equation in previous chapter. Here we are going to find solution of differential equation with different method. It is very useful in different field.

3. SOLUTION OF DIFFERENTIAL EQUATION

General Solutions:-

The general Solution of a differential equation is the most general relation between the dependent and the independent variable occurring in the equation which satisfies the given differential equation.

Particular Solutions:-

Any particular solution that satisfies the given equation is called a particular solution e.g.

$$\frac{dy}{dx} = 5$$

$$\therefore \ dy = 5dx$$

Integrating both sides we get

$$\therefore \int dy = 5 \cdot \int dx + constant$$

Y= 5x+C

This is called as general solution

Suppose C=7 is given

Then particular solution is given by putting of c in the general solution

$$\therefore \ y = 5x+7$$

Check Point:-

1) Find the general solution and particular solution of the differential equation

$\frac{dy}{dx} = x$ When y = 4 at x = 0

Solution: $y = x^2/2 + c$

$$y = x^2/2 + 4$$

Differential equations of first order and of First Degree :-

An equation of the form,

$$M+N \frac{dy}{dx} = 0$$

Where „M" and „N" are functions of x and y or constant. is called differential equation of first order and first degree.

This equation can also be written as

$$Mdx + Ndy = 0$$

7.4 SOLUTION OF DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

There are many methods that can be used to solve the differential equations. Important one among those are listed below.

1) Variable seperable form.
2) Equations reducible to variable seperable form.
3) Homogeneous equations.
4) Exact differential equations.
5) linear differential equations.
6) Equations reducible to linear differential equation.(Bernoullis"s differential equation)

7) Methods of substitution.

We will explain all these methods one by one in detail.

7.4.1 Variable Separable form:-

Working Rule

1)Consider the differential equation Mdx+ Ndy=0

2) If possible rearrange the terms and get f(x) dx +g(y).dy=0

3)Integrate and write constant of integration in suitable form, usually C.

4) Simplify if possible.

Solved Examples:-

Example 1: Solve $\left(3^x \tan y\right)\cdot dx + \left(1-e^x\right)Sec^2 y.dy = 0$

Solution: $\left(3^x \tan y\right)\cdot dx + \left(1-e^x\right)Sec^2 y.dy = 0$

$\div$ throughout by $\left(1\text{-e}^x\right)\cdot \tan y$ we get

$$\left(\frac{3e^x}{1\text{-}e^x}\right)dx + \frac{\sec^2 y}{\tan y}\cdot dy = 0 - - - - - - - - -1)$$

This is in variable separable form

$\therefore$ Integrate equation (1), we get

$$\int \frac{3e^x}{1e^x}\ dx + \int \frac{Sec^2 y}{\tan y}\cdot dy = constant$$

$$\therefore \text{-}3\int \frac{e}{e^x - 1}\cdot dx + \int \frac{Sec^2 y}{\tan y}\cdot dy = c$$

$$\therefore\ \text{-}3\log\left(e^x - 1\right) + \log\ \ \tan y = \log c$$

$$\therefore\ \log\left(e^x - 1\right)^{-3} + \log\ \ \tan y = \log c$$

$$\therefore\ log\left(e^x \text{ - } 1\right)^{-3} \times tany = logc$$

$$\therefore\ \frac{tany}{\left(e^x \text{ - } 1\right)^3} = c$$

$\because$ Removing log both side

$$\therefore\ tany = c \times \left(e^x \text{ - } 1\right)^3$$

This is the general solution of a given differential equation.

Example 2: Solve $\frac{y}{x}\cdot\frac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2 y^2}$

Solution: $\frac{y}{x}\cdot\frac{dy}{dx}=\sqrt{1+x^2+y^2\left(1+x^2\right)}$

$$\frac{y}{x}\times\frac{dy}{dx}=\sqrt{\left(1+x^2\right)\times\left(1+y^2\right)}$$

$$\frac{y}{x}\times\frac{dy}{dx}=\sqrt{\left(1+x^2\right)}\times\sqrt{\left(1+y^2\right)}$$

$$\therefore\ \frac{y}{\sqrt{\left(1+y^2\right)}}\times dy=x\sqrt{\left(1+x^2\right)}\times dx \qquad 1$$

This is in variable separable form

$\therefore$ Integrate equation (1)

$$\frac{1}{2}\cdot\int\frac{2y}{\sqrt{1+y^2}}\cdot dy=\frac{1}{2}\cdot\int 2x\cdot\sqrt{1+x^2}\cdot dx+c$$

$$\left\{\begin{array}{l}\int\frac{f^{1}(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+c\\ \int\left[f(x)\right]^{n}\times f^{1}(x)dx\\ =\frac{\left[f(x)\right]^{n+1}}{n+1}\end{array}\right.$$

$$\therefore\ \frac{1}{2}\times\left[2\sqrt{1+y^2}\right]=\frac{1}{2}\times\left[\frac{2}{3}\times\left(1+x^2\right)^{3/2}\right]+c$$

$$\sqrt{1+y^2}=\frac{1}{3}\left(1+x^2\right)^{3/2}+c$$

This is in required general solution.

Example 3: Solve $(1+x)\cdot\frac{dy}{dx}+1=2e^{-y}$

Solution: The given equation is

$$\therefore\ (1+x)\cdot\frac{dy}{dx}=\left(2e^{-y}-1\right)$$

$$\therefore\ \frac{1}{\left(2e^{-y}-1\right)}\times dy=\frac{1}{x+1}\times dx$$

$$\therefore\ \frac{e^{y}}{e^{y}}\times\frac{1}{\left(2\times e^{-y}-1\right)}\times dy=\frac{1}{x+1}\times dx$$

$$\therefore\ \frac{e^{y}}{2-e^{y}}\cdot dy=\frac{1}{x+1}\cdot dx$$

$$\therefore\ 0=\frac{1}{x+1}\cdot dx-\frac{e^{y}}{2-e^{y}}\cdot dy$$

$$\therefore \frac{1}{x+1}\cdot dx + \frac{e_y}{e^{y-2}}\cdot dy = 0 - - - - - (1)$$

This is in variable separable form,

Integrate equation (1), we get

$$\int \frac{1}{x+1}\cdot dx + \int \frac{e^y}{\left(e^y - 2\right)}\, dy = \log c$$

$$\therefore \log(x+1) + \log\left(e^y - 2\right) = \log c$$

$$\therefore \log\left[(x+1)\cdot\left(e^y - 2\right)\right] = \log c$$

$$\therefore (x+1)\cdot\left(e^y - 2\right) = c$$

This is the required general solution.

Example 4: Solve $3e^x \tan y \cdot dx + \left(1+e^x\right)\sec^2 y \cdot dy = 0$

given $y = \frac{®}{4}$ when x=0

Solution: The given equation is

$$3e^x \tan y \cdot dx + \left(1+e^x\right)\sec^2 y \cdot dy = 0$$

÷ through out by $\left(1+e^x\right)\cdot \tan y$

$$\therefore \frac{3e^x}{1+e^x}\cdot dx + \frac{\sec^2 y}{\tan y}\cdot dy = 0 - - - - - - - - - - -(1)$$

This is in variable separable form,

Integrate equation (1) we get

$$\therefore \int \frac{3e^x}{1+e^x}\cdot dx + \int \frac{\sec^2 y}{\tan y}\cdot dy = \log c$$

$$\therefore 3\int \frac{e^x}{1+e^x}\cdot dx + \int \frac{\sec^2 y}{\tan y}\cdot dy = \log c \otimes$$

$$3\log\left(1+e^x\right) + \log\ \tan y = \log\ c$$

$$\therefore \log\left(1+e^x\right)^3 + \log\ \tan y = \log\ c$$

$$\therefore \log\left[\left(1+e^x\right)^3\cdot \tan y\right] = \log c$$

$$\therefore \left(1+e^x\right)^3\cdot \tan y = c ---------------(2)$$

This is the required general solution

To final particular solution:-

put y=®⁄4 at x=0 in equation --------(2)

$\therefore\ (1+1)^3 \cdot \tan\ ®/4 = c$

$\therefore$ c=8

Put value of c in equation (2)

$$\therefore (1+e^x)^3 \cdot \tan y = 8$$

This is a particular solution

Example 5: Solve $\dfrac{dy}{dx} = \dfrac{x\ (2\log x\ +1)}{\sin\ y + \ y\cos y}$

Solution: The given equation is

$$\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$$

$$\therefore\ (\sin y + y\cos y)\cdot dy = x(2\log x+1)\cdot dx - - - - - - - (1)$$

This is in variable separable form

Integrate equation (1), we get

$$\int(\sin y + y\cos y)\cdot dy = \int x(2\log x+1)\cdot dx + cons\tan t$$

$$\therefore\ \int \text{siny}\cdot dy + \int y\cdot\cos y\cdot dy = 2\int x\cdot\log x\cdot dx + \int xdx + c$$

$$-\cos y + y\sin y + \cos y = 2\cdot\left[\log x\cdot\frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2}\right] + c$$

$$y\sin y = x^2\log x - x^2 + x^2 + c$$

$$\therefore\ y\sin y = x^2\log x + c$$

This is required general solution

Check Your Progress:

1) solve: $\dfrac{dy}{dx} = e^{x-y} + x^2\cdot e^{-y}$

$e^x + \dfrac{x^3}{3} - e^y = c$

2) solve: $\left(y - x\cdot\dfrac{dy}{dx}\right) = a\cdot\left(y^2 + \dfrac{dy}{dx}\right)$

ans $(1-ay)(x+a) = cy$

3) solve: $\log\dfrac{dy}{dx} = ax + by$

ans $\dfrac{e^{ax}}{a} + \dfrac{e^{-by}}{b} = c$

4) solve: $x\cos x\cos y + \sin y\cdot\dfrac{dy}{dx} = 0$

ans $x\sin x + \cos x - \log\cos y = c$

5) solve: $Sec^2 x\cdot \tan y\cdot dx+\sec^2 y\cdot \tan x\cdot dy=0$

ans $\tan x\cdot \tan y=c$

6) solve: $\frac{dy}{dx}=e^{x-2y}$

ans $\frac{1}{2}\cdot e^{2y}-e^{x}=c$

7.4.2 Equations Reducible to variable separable forms:

Sometimes we come across differential equations which cannot be converted into variable separable form by mere rearrangement of its terms. These differential equation can be suitable substitution

Solved Examples:-

Example 6: solve: $(x-y)^2\cdot \frac{dy}{dx}=a^2$

Solution: we have $(x-y)^2\cdot \frac{dy}{dx}=a^2$ ----------(1)

Substitute x-y=t

Differentiating with respect to x, we get

$$1-\frac{dy}{dx}=\frac{dt}{dx}$$

$$\therefore \frac{dy}{dx}=1-\frac{dt}{dx}$$

Using equation (1) we have

$$t^2\cdot\left(1-\frac{dt}{dx}\right)=a^2$$

$$\therefore 1-\frac{dt}{dx}=\frac{a^2}{t^2}$$

$$\therefore \frac{dt}{dx}=1-\frac{a^2}{t^2}$$

$$\therefore \frac{dt}{dx}=\frac{t^2-a^2}{t^2}$$

$$\therefore \frac{t^2}{t^2-a^2}\cdot dt=dx$$

This is invariable separable form

Integrating we get

$$\int dx=\int\frac{t^2}{t^2-a^2}\cdot dt+cons\tan t$$

$$\therefore x=\int\frac{t^2-a^2+a^2}{t^2-a^2}\cdot dt+c$$

$$\therefore\ x=\int dt+\int\frac{a^2}{t^2-a^2}\cdot dt+c$$

$$\therefore\ x=t+a^2\cdot\frac{1}{2a}\cdot\log\left(\frac{t-a}{t+a}\right)+c$$

$$\therefore\ x=t+\frac{a}{2}\cdot\log\left(\frac{t-a}{t+a}\right)+c$$

t=x-y

$$\therefore\ x=x-y+\frac{a}{2}\cdot\log\left(\frac{x-y-a}{x-y+a}\right)+c$$

$$y=\frac{a}{2}\cdot\log\left(\frac{x-y-a}{x-y+a}\right)+c$$

This is the required general solution

Example 7: Solve $\frac{dy}{dx}=\cos\left(x+y\right)$

Solution: We have $\frac{dy}{dx}=\cos\left(x+y\right)----------(1)$

Put $x+y=t$

Differentiating above with respect to x, we get

$$\therefore\ 1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\therefore\ \frac{dy}{dx}=\frac{dt}{dx}-1$$

Using equation (1)

$$\therefore\ \frac{dt}{dx}-1=\cos t$$

$$\therefore\ \frac{dt}{dx}=1+\cos t$$

$$\therefore\ \frac{1}{1+\cos t}\cdot dt=dx$$

$$\therefore\ \frac{1}{2\cos^2 t/2}dt=dx$$

This is invariable separable form,

Integrating we get

$$\therefore\ \int\frac{1}{2\cos^2 t/2}\cdot dt=\int dx+constant$$

$$\therefore\ \frac{1}{2}\cdot\int\sec^2 t/2\cdot dt=x+c$$

$$\therefore \frac{1}{2}\cdot\frac{2}{1}\cdot\tan^{-1}\frac{t}{2} = x+c$$

$$\therefore \tan^{-1}\frac{t}{2} = x+c$$

t=x+y

$$\therefore \tan^{-1}\left(\frac{x+y}{2}\right) = x+c$$

This is the required general solution,

Example 8: Solve $(4x+y)^2\cdot\frac{dx}{dy}=1$

Solution: The given equation is $\frac{dy}{dx}=(4x+y)^2$ ------(1)

Put $(4x+y)=t$

Differentiating above with respect to x

$$\therefore 4+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{dt}{dx}-4$$

Using equation (1), we have

$$\frac{dt}{dx}-4=t^2$$

$$\therefore \frac{dt}{dx}=t^2+4$$

$$\therefore \frac{1}{t^2+4}\cdot dt = dx$$

This is in variable separable form

Integrating we get,

$$\therefore \int\frac{1}{t^2+4}\cdot dt=\int dx+\text{constant}$$

$$\therefore \frac{1}{2}\cdot\tan^{-1}\left(\frac{t}{2}\right)=x+c$$

t=x+y

$$\therefore \frac{1}{2}\cdot\tan^{-1}\left(\frac{x+y}{2}\right)=x+c$$

$$\therefore \tan^{-1}\left(\frac{x+y}{2}\right)=2x+c_1 \text{ where } c_1=c$$

This is the required general solution

Example 9: Solve $(x+y)\cdot\frac{dy}{dx}+y=0$

Solution:

$$(x+y)\cdot\frac{dy}{dx}+y=0----------(1)$$

Put $x+y=t$

Differentiating with respect to x, we get

$$\therefore\ 1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\therefore\ \frac{dy}{dx}=\frac{dt}{dx}-1$$

Using equation (1), we have

$$\therefore\ t\cdot\left(\frac{dt}{dx}-1\right)+t-x=0$$

$$\frac{dt}{dx}-1=\frac{x-t}{t}$$

$$\therefore\ \frac{dt}{dx}-1=\frac{x}{t}-1$$

$$\therefore\ \frac{dt}{dx}=\frac{x}{t}$$

$$xdx=tdt$$

This is in variable separable form

Integrating we get,

$$\int xdx=\int tdt+\text{constant}$$

$$\frac{x^2}{2}=\frac{t^2}{2}+c$$

$$\therefore\ x^2=t^2+2c$$

t=x+y

$$\therefore\ x^2=(x+y)^2+2c$$

$$\therefore\ x^2=x^2+2xy+y^2+2c$$

$$\therefore\ 2xy+y^2=-2c$$

$$\therefore\ y^2+2xy=c_1 \text{ where } c_1=-2c$$

This is the required general solution

Example 10: Solve $\left(\frac{y}{x}\cos\frac{y}{x}\right)\cdot dx-\left(\frac{x}{y}\cdot\sin\frac{y}{x}+\cos\frac{y}{x}\right)\cdot dy=0$

Solution:

The equation is, $\left(\frac{y}{x}\cos\frac{y}{x}\right)\cdot dx-\left(\frac{x}{y}\cdot\sin\frac{y}{x}+\cos\frac{y}{x}\right)\cdot dy=0$

Substitute $\frac{y}{x} = v$

$\therefore$ y=vx

Differentiating above with respect to x, we get

$$\therefore \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

But the above equation can be written as

$$\therefore \frac{y}{x} \cdot \cos\frac{y}{x} - \left(\frac{x}{y} \cdot \sin\frac{y}{x} + \cos\frac{y}{x}\right) \cdot \frac{dy}{dx} = 0$$

$$\therefore \text{ v cosv-}\left(\frac{1}{v} \cdot \sin v + \cos v\right) \cdot \left(v + x \cdot \frac{dy}{dx}\right) = 0$$

By rearranging the terms, we have

$$\therefore \frac{1}{x} \cdot dx = -\frac{\sin v + v\cos v}{v\sin v} dv$$

$$\therefore \frac{1}{x} \cdot dx + \frac{\sin v + v\cos v}{v\sin v} dv = 0$$

This is in variable separable form

Integrating we get,

$$\therefore \int \frac{1}{x} \cdot dx + \int \frac{\sin v + v\cos v}{v\sin v} \cdot dv = cons\tan t$$

$$\therefore \text{ logx+log}(\text{vsinv}) = c$$

$$\text{log}(\text{x} \cdot \text{vsinv}) = \text{log}\, c$$

$$\text{xv} \cdot \text{sinv=c}$$

$$\text{v=}\frac{y}{x}$$

$$\therefore \text{ x} \cdot \frac{y}{x} \sin\frac{y}{x} = c$$

$$\therefore \text{ ysin } \frac{y}{x} = c$$

This is the required general solution

Check Your Progress:

Solve the following

1) $\frac{dy}{dx} + e^{y/x} = \frac{y}{x}$ Ans : log cx=$e^{-y/x}$

2) $\left(1 + e^{x/y}\right) + e^{x/y}\left(1 - \frac{x}{y}\right) \cdot \frac{dy}{dx} = 0$ Ans : $\text{x+y} \cdot \text{e}^{x/y} = c$

3) $(2x-y)\cdot e^{y/x}+\left(y+x\cdot e^{y/x}\right)\cdot\frac{dy}{dx}=0$ Ans: $y^2+2x^2e^{y/x}=c$

$\left[\tan\frac{y}{x}-\frac{y}{x}\cdot\sec^2\frac{y}{x}\right]dx+\sec^2\frac{y}{x}\cdot dy=0$

Ans $x+\tan\left(\frac{y}{x}\right)=c$

7.4.3 Homogeneous Equations

A differential equation Mdx+Ndy=0 is said to be homogeneous if M & N are homogeneous functions of x and y of same degree

Working Rule:

1) Check whether differential equation is homogenous in x and y
2) Express $\frac{dy}{dx}$ in terms of x and y
3) Put y=vx
4) $\therefore \frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$
5) Separate x and y variables and get F(ex)dx+ g(v) dv=0
6) Solve by integration
7) Put $v=\frac{y}{x}$ and simplify

Solved examples:-

Example 11: Solve $\left(x^2+y^2\right)dx+2xy\cdot dy=0$

Solution: We have $\left(x^2+y^2\right)dx+2xy\cdot dy=0$

Here M and N are homogeneous expressions in x and y of the second degree

$\therefore$ 2xy·dy=-$\left(x^2+y^2\right)dx$

$\therefore$ 2xy·dy=-$\left(x^2+y^2\right)dx$

$$\therefore \frac{dy}{dx}=-\cdot\left(\frac{x^2+y^2}{2xy}\right)------------(1)$$

put y=vx

$$\therefore \frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$$

Using equation 1 we have

$$v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{-2x\cdot vx}$$

$$\therefore\ v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{-2v\cdot x^2}$$

$$\therefore\ x\frac{dv}{dx} = \frac{1+2v^2}{-2v} - 1$$

$$\therefore\ x\frac{dv}{dx} = \frac{1+3v^2}{-2v}$$

$$\therefore\ \frac{-2v}{1+3v^2}\cdot dv = \frac{1}{x}dx$$

This is in variable separable form

Integrating above expression we have

$$\therefore\ -\frac{1}{3}\int\frac{6v}{1+3v^2}\cdot dv = \int\frac{1}{x}dx + constant$$

$$\therefore\ -\frac{1}{3}\log(1+3v^2) = \log x + \log c$$

$$\therefore\ -\frac{1}{3}\log(1+3v^2) = \log(cx)$$

$$\therefore\ \log(1+3v^2) = -3\log(cx)$$

$$\therefore\ \log(1+3v^2) = -3\log(cx)^{-3}$$

$$\therefore\ 1+3v^2 = \frac{1}{c^3x^3}$$

$$v = \frac{y}{x}$$

$$\therefore\ 1+3\cdot\frac{y^2}{x^2} = \frac{1}{c^3x^3}$$

$$\therefore\ x^3 + 3xy^2 = \frac{1}{c^3}$$

$$\therefore\ x^3 + 3xy^2 = k \text{ where } k = \frac{1}{c^3}$$

This is the required general solution

Example 12: Solve $y^2 + x^2\cdot\frac{dy}{dx} = xy\cdot\frac{dy}{dx}$

Solution:

The given equation is $y^2 + x^2\cdot\frac{dy}{dx} = xy\cdot\frac{dy}{dx}$

$$\therefore\ y^2 = xy\cdot\frac{dy}{dx} - x^2\cdot\frac{dy}{dx}$$

$$\therefore\ y^2 = \frac{dy}{dx}(xy - x^2)$$

$$\therefore \frac{dy}{dx}=\frac{y^2}{xy-x^2}---------(1)$$

This is a homogeneous equation

Put y=vx

$$\therefore \frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$$

Using equation (1),we have

$$\therefore v+x\cdot\frac{dv}{dx}=\frac{v^2x^2}{x\cdot vx-x^2}$$

$$\therefore v+x\cdot\frac{dv}{dx}=\frac{v^2x^2}{x^2(v-1)}$$

$$\therefore x\cdot\frac{dv}{dx}=\frac{v^2}{(v-1)}-v$$

$$\therefore x\cdot\frac{dv}{dx}=\frac{v}{(v-1)}$$

$$\therefore \frac{v-1}{v}\cdot dv=\frac{1}{x}\cdot dx$$

$$\therefore \left(1-\frac{1}{v}\right)dv=\frac{1}{x}dx$$

This is in variable separable form

Integrating we get,

$$\int\left(1-\frac{1}{v}\right)\mathrm{dv}=\int\frac{1}{\mathrm{x}}dx+constant$$

$$\therefore \mathrm{vlogv=logx+logc}$$

$$\therefore \mathrm{v=logv+logx+logc}$$

$$\therefore \mathrm{v=log(vxc)}$$

$$\mathrm{v}=\frac{\mathrm{y}}{\mathrm{x}}$$

$$\therefore \frac{\mathrm{y}}{\mathrm{x}}=\log\left(\frac{\mathrm{y}}{x}\cdot x\cdot c\right)$$

$$\therefore \frac{\mathrm{y}}{x}=\log cy$$

$$\therefore \mathrm{y=xlog}\ cy$$

This is the required general solution

Example 13: solve $(x^3+y^3)dx-3xy^2\cdot dy=0$.

Solution:

$$\left(x^3 + y^3\right)dx - 3xy^2 \cdot dy = 0$$

This is a homogenous equation

$$\left(x^3 + y^3\right)dx = 3xy^2 \cdot dy$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{x^3 + y^3}{3xy^2} -------- (1)$$

Put $y = vx$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{dv}{dx}$$

Using equation 1 we have

$$\therefore\ v + x\frac{dv}{dx} = \frac{x^3 + v^3x^3}{3x \cdot v^2x^2}$$

$$\therefore\ v + x\frac{dv}{dx} = \frac{x^3\left(1 + v^3\right)}{3v^2 \cdot x^3}$$

$$\therefore\ x\frac{dv}{dx} = \frac{1 + v^3}{3v^2} - v$$

$$\therefore\ x\frac{dv}{dx} = \frac{1 - 2v^3}{3v^2} -$$

$$\therefore\ \frac{3v^2}{1 - 2v^3} \cdot dv = \frac{1}{x}dx$$

This is in variable separable form

Integrating we have

$$-\frac{1}{2} \cdot \int \frac{6v^2}{2v^3 - 1} \cdot dv = \int \frac{1}{x}\, dx + cons\tan t$$

$$\therefore\ -\frac{1}{2}\log\left(2v^3 - 1\right) = \log x + \log c$$

$$\therefore\ \log\left(2v^3 - 1\right) = -2\log\left(cx\right)$$

$$\therefore\ \not{l}o\not{g}\left(2v^3 - 1\right) = \not{l}o\not{g}\left(cx\right)^{-2}$$

$$\therefore\ \left(2v^3 - 1\right) = \frac{1}{c^2x^2}$$

Put $v = \frac{y}{x}$

$$\therefore\ 2\frac{y^3}{x^3} - 1 = \frac{1}{c^2x^2}$$

$$\therefore\ 2y^3 - x^3 = \frac{x}{c^2}$$

$$\therefore\ 2y^3 - x^3 = kx \text{ where } k = \frac{1}{c^2}$$

This is the required general solution

Example 14: solve $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\cdot dy = 0$

Solution:

The given equation is

$$\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\cdot dy = 0$$

$$\therefore\ \frac{dy}{dx} = \frac{y\sec^2\frac{y}{x} - x\tan\frac{y}{x}}{x\sec^2\frac{y}{x}}$$

$$\therefore\ \frac{dy}{dx} = \frac{y}{x} - \frac{\tan\frac{y}{x}}{\sec^2\frac{y}{x}} ---------(1)$$

This is a homogeneous equation

Put $y = vx$

$$\therefore\ \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$$

Using equation 1 we have

$$\therefore\ \not{v} + x\cdot\frac{dv}{dx} = \not{v} - \frac{\tan v}{\sec^2 v}$$

$$\therefore\ x\cdot\frac{dv}{dx} = \frac{\tan v}{\sec^2 v}$$

$$\therefore\ \frac{\sec^2 v}{\tan v}\cdot dv = -\frac{1}{x}\cdot dx$$

$$\therefore\ \frac{\sec^2 v}{\tan v}\cdot dv + \frac{1}{x}\cdot dx = 0$$

This is in variable separable form

Integrating we get,

$$\therefore\ \int\frac{\sec^2 v}{\tan^2 v}\cdot dv + \int\frac{1}{x}dx = constant$$

$\therefore$ log tanv + log x = log c

$\therefore$ log (tanv · x) = log c

$\therefore$ x · tanv = c

Put $v = \frac{y}{x}$

$\therefore\ x \cdot \tan\frac{y}{x} = c$

This is the required general solution

Check Your Progress:

1) solve the following

i) $xdy - ydx = \sqrt{x^2 + y^2} \cdot dx$

ans $y + \sqrt{x^2 + y^2} = cx^2$

ii) $\left(x + y \cdot \cot\frac{x}{y}\right)dy - ydx = 0$

ans $y = c\sec\frac{x}{y}$

iii) $y^2 + x^2 \cdot \frac{dy}{dx} = xy \cdot \frac{dy}{dx}$

ans $cy = e^{y/x}$

iv) $(x^2 - y^2)dx = 2xydy$

ans $x(x^2 - 3y^2) = c$

v) $x\frac{dy}{dx} = y + \sqrt{x^2 + a^2}$

ans $y = c \cdot e^{x^2/3y^2}$

vi) $(x + y) \cdot \frac{dy}{dx} = x - y$

ans $-y^2 - 2xy + x^2 = c$

7.4.4 Exact Differential Equation

Definition:-

The equation Mdx+Ndy=0 is said to be an exact differential equation if and only it.

Mdx+ Ndy=du

Where u is some function of x and y

e.g. xdy+ydx=0 is exact

$\because$ u=xy

Where

xdy+ydy = du

Necessary and sufficient condition :-

The necessary and sufficient condition that the equation Mdx+Ndy=0 is exact is.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Rules for the General solution:-

If the equation Mdx+Ndy=0 is exact then its general solution is given by

$$\int M \ (treat \text{ y as constant}) \ dx + \int N \ (terms \text{ free from x }) \ dy = c$$

Where

(1) In first integral with respect to x, treat y as constant

(ii)In second integral do not take the terms containing x i.e. take only those terms of N which are free from x. If no such term is available then second integrals may not be considered.

(iii) c is arbitrary constant of Integration.

Solved Examples:-

Example15: Solve $\left(5x^4+6x^2y^2-8xy\right) dx+\left(4x^3y-12x^2y^2-5y^4\right)\cdot dy=0$

Solution: The given equation is:

$$\left(5x^4+6x^2y^2-8xy^3\right)dx+\left(4x^3y-12x^2y^2-5y^4\right)dy=0------(1)$$

$\therefore$ M= $5x^4+6x^2y^2-8xy^3$

N= $4x^3y-12x^2y^2-5y^4$

$$\therefore \frac{\partial \text{M}}{\partial \text{Y}} = \frac{\partial}{\partial Y}\left(5x^4+6x^2y^2-8xy^3\right)$$

$$= 0+12x^2y-24xy^2$$

$$\therefore \frac{\partial \text{M}}{\partial \text{Y}} = 12x^2y-24xy^2$$

$$\therefore \frac{\partial \text{N}}{\partial \text{x}} = \frac{\partial}{\partial \text{x}}\left(4x^3y-12x^2y^2-5y^4\right)$$

$$= 12x^2y-24xy^2-0$$

$$\therefore \frac{\partial \text{N}}{\partial \text{x}} = 12x^2y-24xy^2$$

$$\therefore \frac{\partial M}{\partial \text{y}} = \frac{\partial N}{\partial \text{x}}$$

Hence differential equation (1) is exact

Its solution is given by

$$\int M\left(treat \text{ y constant}\right)dx+\int N\left(terms \text{ free from x}\right)\cdot dy=c$$

$$\therefore \int (5x^4+6x^2y^2-8xy^3)dx+\int(-5y^4)\cdot dy=c$$

$$\therefore 5\cdot\frac{x^5}{5}+6^2y\cdot\frac{x^3}{3}-8^4y^3\cdot\frac{x^2}{2}-5\cdot\frac{y^5}{5}=c$$

$$x^5+2x^3y^2-4x^2y^3-y^5=c$$

This is the required general solution

Example 16: Solve $\frac{dy}{dx}=-\frac{4x^3y^2+y\cos xy}{2x^4y+x\cos xy}$

Solution**:**

The given equation is

$$\frac{dy}{dx}=-\frac{4x^3y^3+y\cos xy}{2x^4y+x\cos xy}$$

$$\therefore (4x^3y^2+y\cos xy)\,dx+(2x^4+y\cos xy)\,dy=0..........(1)$$

Comparing with Mdx+Ndy=0; we have

$$M=4x^3y^2+y\cos xy$$

$$N=2x^4y+x\cos xy$$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(4x^3y^2+y\cos xy)$$

$$\frac{\partial M}{\partial y}=8x^3y^2+\cos xy-xy\sin xy$$

$$\therefore \frac{\partial N}{\partial x}=\frac{\partial}{x}(2x^4y\ +y\cos xy)$$

$$\frac{\partial N}{\partial x}=8x^3y\ +\cos xy-xy\sin xy$$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Hence differential equation (1) is exact

Its solution is given by

$$\int Min(treat\ \text{y constant})dx+\int N(terms\ \text{free from x})\cdot dy=c$$

$$(4x^3y^2+y\cos xy)\ dx+\int o dy=c$$

$$4y^2\int x^3dx+y\int\cos\ xy=c$$

$$4y^2\cdot\frac{x^4}{4}+y\frac{\sin xy}{y}=c$$

$$\therefore\quad x^4y^2+\sin xy=c$$

This is the required general solution

Example 17: Solve $(x-2e^y)dy+(y+x\sin x)dx=0$.

Solution:

The equation given is

$$\left(x-2e^{y}\right)dy+\left(y+x\sin x\right)dx=0$$

$$\therefore\ \left(\text{y+xsinx}\right)dx+\left(x-2e^{y}\right)dy=0------(1)$$

Comparing with Mdx+Ndy=0; we have

$$M=y+x\sin x$$

$$N=x-2e^{y}$$

$$\therefore\ \frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(y+\sin x\right)$$

$$\therefore\ \frac{\partial M}{\partial y}=1$$

$$\therefore\ \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(x-2e^{y}\right)$$

$$\therefore\ \frac{\partial N}{\partial x}=1$$

$$\therefore\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Hence differential equation (1) is exact

Its solution is given by

$$\int M\left(treat\ \text{y constant}\right)dx+\int N\left(terms\ \text{free from x}\right)\cdot dy=c$$

$$\therefore\ \int\left(\text{y+xsinx}\right)dx+\int\left(-2\cdot e^{y}\right)\cdot dy=c$$

$$\therefore xy+\left[x\left(-\cos x\right)+\sin x\right]-2\cdot e^{y}=c$$

This is the required general solution

Example 18: Solve

$$\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left(x+\log x-x\sin y\right)\text{dy}=0$$

Solution: The given equation is

$$\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left(x+\log x-x\sin y\right)\cdot dy=0----(1)$$

Comparing with Mdx+Ndy=0; we have

$$M=y\left(1+\frac{1}{x}\right)+\cos y$$

$$N=x+\log x-x\sin y$$

$$\therefore\ \frac{\partial M}{\partial Y}=\frac{\partial}{\partial y}\left(y\left(1+\frac{1}{x}\right)+\cos y\right)$$

$$\frac{\partial M}{\partial Y} = 1 + \frac{1}{x} - \sin y$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \cdot (x + \log x - x \sin y)$$

$$\therefore \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial Y} = \frac{\partial N}{\partial x}$$

Hence the differential equation (1) is exact

Its solution is given by

$$\int M \,(treat \text{ y constant})\, dx + \int N \,(\text{terms free from x})\, dy = c$$

$$\int \left(y \cdot \left(1 + \frac{1}{x} \right) + \cos y \right) dx + \int o\, dy = c$$

$$y \cdot \int \left(1 + \frac{1}{x} \right) dx + \int \cos y \, dy = c$$

$$y(x + \log x) + x \cos y = c$$

This is the required general solution

Example 19: Solve $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$

Solution: The given equation is

$$\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{(y \cos x + \sin y + y)}{(\sin x + x \cos y + x)}$$

$$\therefore (\sin x + x \cos y + x) dy = -(y \cos x + \sin y + y) dx$$

$$\therefore (y \cos x + \sin y + y) dx + (\sin x + x \cos y + x) dy = 0 ----(1)$$

Comparing with Mdx+Ndy=0; we have

M=$y \cos x + \sin y + y$

N=$\sin x + x \cos y + x$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (y \cos x + \sin y + y)$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (\sin x + x \cos y + x)$$

$$\therefore \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the differential equation (1) is exact

Its solution is given by

$$\int M\left(treat \text{ y constant}\right)dx + \int N\left(terms \text{ free from x}\right)dy = c$$

$$\therefore \quad \int \left[y\cos x + \sin y + y\right] dx + \int o dy = c$$

$$\therefore \quad \text{y} \cdot \int y\cos x \cdot dx + \sin y \cdot \int dx + y \cdot \int dx = c$$

$$\therefore \quad y\sin x + x\sin y + xy = c$$

Which is the require general solution

Check Your Progress:

Solve:(1) $\left(a^2 - 2xy - y^2\right)dx - \left(x+y\right)^2 \cdot dy = 0$

Ans. $a^2x - x^2y - xy^2 - \frac{y^3}{3} = c$

(2) $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Ans. $x + y \cdot e^{x/y} = c$

(3) $\left[\cos x \cdot \tan y + \cos\left(x+y\right)\right]dx + \left[\sin x \cdot \sec^2 y + \cos\left(x+y\right)\right]dy = o$

Ans. $\sin x \cdot \tan y + \sin\left(x+y\right) = c$

(4) $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xy \cdot e^{xy^2} - 3y^2\right)dy = 0$

Ans. $e^{xy^2} + x^4 - y^3 = c$

(5) $\left[1 + \log\left(xy\right)\right]dx + \left\{1 + \frac{x}{y}\right\}dy = 0$

Ans. $y + x\log\left(xy\right) = c$

(6) $\left(2xy + e^y\right)dx + \left(x^2 + xe^y\right) \cdot dy = 0$

Ans. $x^2y + xe^y = c$

(7) $\left[y\sin\left(xy\right) + xy^2\cos\left(xy\right)\right]dx + \left[x\sin\left(xy\right) + x^2y\cos\left(xy\right)\right]dy = 0$

Ans. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \sin xy + xy\cos(xy) + 2xy\cos(xy) - x^2y^2\sin(xy)$

General solution is given by

$$xy\sin\left(xy\right) = c$$

7.5 LET US SUM UP

In this chapter we have learn

- solution of D.E:- general solution, particular solution
- variable separable form:- dx

$\int f(x)dx = \int f(y)dy + c$

- Equations reducible to variable separable form.
- Homogeneous differential equation i.e $\frac{dy}{dx} = \frac{f(xy)}{g(xy)}$

With substituting Y=Yx.

7.6 UNIT END EXERCISE

Solve the following differential equation.

i. $\frac{dy}{dx} = \frac{sinx + xcosx}{Y(1+2logu)}$

ii. $\frac{dy}{dx} + x^2 = x^2e^3y$

iii. $2x\,Cosy\,dx - (1+x^2)siny\,dy = 0$

iv. $(x+1)\frac{dy}{dx} + 1 = e^{-2y}$

v. $\frac{dy}{dx} = ax + by + c$

vi. $\frac{dy}{dx} = sin(x+y) + cos(x+y)$

vii. $\frac{dy}{dx} = e^{y/x} + y/x$

viii. $\frac{dy}{dx} = (4x+y+1)^2$

ix. $\frac{dy}{dx} = y/x + sin(y/x)$

x. $\frac{dy}{dx} = (x+y+1)^2$

xi. $\frac{dy}{dx} = 1 + y/x - Cos\, y/x$

xii. $(x^3 + y^3)\frac{dy}{dx} = x^2y$

xiii. $\left(4 - \frac{y^2}{x^2}\right)dx + \frac{2y}{x}dy = 0$

xiv. $$\frac{dy}{dx}+\frac{x^2+3y^2}{3x^2+y^2}=0$$

xv. $$4(x+y)\frac{dy}{dx}=3x-4y$$

*****

8

EQUATION REDUCIBLE TO EXACT EQUATIONS

UNIT STRUCTURE

1. Objective
2. Introduction
3. Definition
4. Linear Equation And Equations Reducible To Linear Form
5. Equations reducible to linear form
6. Let Us Sum Up
7. Check your progress
8. Unit End Exercise

8.1OBJECTIVE

After going through this chapter you will able to

- Find the solution of non-exact .differential equation.
- Find the solution of linear .differential equation.
- Reducing to non-linear equation into linear
- equation. Find the solution of non-linear equation.

8.2 INTRODUCTION

In previous chapter we have learn about exact differential equation & its solution. Now here we are going to discuss none exact differential equation. To find the solution of non-exact differential equation we use integrating factor which convert non-exact differential equation to exact differential equation. Also we discuss about solution of linear differential equation.

In some cases equations which are not exact can be converted to exact differential equation by multiplying by some suitable factor called as Integrating factor.

8.3 DEFINITION

Integrating Factor

If the equation leMdx +leNdy=0 is exact

then le is said to be an integrating factor of the equation Mdx+ Ndy = 0

8.3.1 Rules of finding Integrating factor :-

Rule (1)

If the equation Mdx+Ndy=0 is homogeneous then $\frac{1}{Mx+Ny}$ is integrating factor

Solved Example:

Example 1: $\left(x^2y-2xy^2\right)$ dx-$\left(x^3-3x^2y\right)dy=0$

Solution: The given equation is

$\left(x^2y-2xy^2\right)$ dx-$\left(x^3-3x^2y\right)dy=0$........(1)

This is a homogeneous equation.

Comparing with Mdx +Ndy=0 ; we have

$$M=x^2y-2xy^2$$

$$N=-\left(x^3-3x^2y\right)$$

$$\therefore \text{ I.f. } =\frac{1}{Mx+Ny}$$

$$=\frac{1}{x^3y-2x^2y-x^3y+3x^2y^2}$$

$$\therefore \frac{\left(x^2y-2xy^2\right)}{x^2y^2}\,dx-\frac{\left(x^3-3x^2y\right)}{x^2y^2}\,dy=0 \text{ is exact}$$

$$\text{i.e. } \left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0 \text{ is exact}$$

Its general solution is given by

$$\int M\ (\text{treat y const})\ dx+\int N\ (\textit{terms free from } x)\ dy=c$$

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

$$\therefore \frac{1}{y}\cdot\int dx-2\int\frac{1}{x}dx+3\cdot\int\frac{1}{y}dy=c$$

$$\therefore \frac{x}{2}-2\log x+3\log y=c$$

This is required general solution

Check your progress:

Solve

i) $\left(3xy^2-y^3\right)dx+\left(xy^2-2x^2y\right)dy=0$

Hint : I.F. $=\dfrac{1}{x^2y^2}$

General solution is given by

$$\frac{cy^2}{x^3}=c^{y/x}$$

ii) $\left(x^2-3xy+2y^2\right)dx+x\left(3x\text{-}2y\right)dy=0$

Hint: I.F. $=\dfrac{1}{x^3}$

General solution is given by

$x^2\log x+3xy=y^2+cx^2$

8.3.2 Rule (II) :

If the equation Mdx+Ndy=0 can be written as

$M=y\,f_1(xy)dx,\quad N=xf_2(xy)\cdot dy=0$

i.e. $M=y\,f_1\,(xy),\quad N=x\,f_2\,(xy)$

then $\dfrac{1}{Mx\text{-}Ny}$ is an integration factor.

Note :- $f_1\,(x\,y),\ f_2\,(x\,y)$ are functions of xy.

Solved Examples :-

Example 2: Solve $\left(x^2y^2+2\right)ydx+\left(2-2x^2y^2\right)xdy=0$

Solution**:** The equation is given by

$$\left(x^2y^2+2\right)ydx+\left(2-2x^2y^2\right)xdy=0$$

Comparing with Mdx+Ndy=0; we have

$\therefore\ M = (x^2y^2 + 2)y$

$N = (2\text{-}2x^2y^2)\cdot x$

$I.f. = \frac{1}{Mx - Ny}$

$\therefore\ I.f. = \frac{1}{xy(x^2y^2 + 2 - 2 + 2x^2y^2)}$

$I.f. = \frac{1}{3x^3y^3}$

$\therefore\ \frac{(x^2y^2+2)y}{3x^3y^3}dx + \frac{(2\text{-}2x^2y^2)\cdot x}{3x^3y^3}dy = 0$

$i.e \left(\frac{1}{3x} + \frac{2}{3}\cdot\frac{1}{x^3y^2}\right) dx + \left(\frac{2}{3x^3y^3} - \frac{2}{3y}\right)\cdot dy = 0$

which is a exact equation

$\therefore$ Its General solution is given by

$\int M\ (treat\ y\ constant)\ dx + \int N\ (terms\ free\ from\ x)\ dy = c$

$\therefore\ \int\left(\frac{1}{3x} + \frac{2}{3}\cdot\frac{1}{x^3y^2}\right) dx + \int -\frac{2}{3y}\cdot dy = c$

$\therefore\ \frac{1}{3}\int\frac{1}{x}dx + \frac{2}{3y^2}\cdot\int\frac{1}{x^3}dx - \frac{2}{3}\int\frac{1}{y}\cdot dy = c$

$\therefore\ \frac{1}{3}\log x - \frac{2}{6x^2y^2} - \frac{2}{3}\log y = c$

$\therefore\ \log x - \frac{1}{x^2y^2} - 2\log y = c_1$ where $c_1 = 2c$

Check your progress:

1. solve :

$(x^2y^2 + xy + 1)\ y\cdot dx + (x^2 + y^2 - xy + 1)\ xdy = 0$

Hint: I.F. $\frac{1}{2x^2y^2}$

$G.S.\ is$ given by

$xy + \log x - \frac{1}{xy} - \log y = c$

2. $y(xy + 2x^2y^2) + x\ (xy\text{-}x^2y^2)\ dy = 0$

Ans $x^2 = cy\cdot e^{-\frac{1}{xy}}$

8.3.3 Rule (III):

If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = a$ function of x alone. Say f(x) then $e\int f(x)\,dx$ is integrated. factor.

Solved Examples :-

Example 3: Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

Solution: The given equation is

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$$

Comparing with Mdx+Ndy=0; we get

$$M = y^4 + 2y$$

$$N = xy^3 + 2y^4 - 4x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y^4 + 2y\right)$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(xy^3 + 2y^4 - 4x\right)$$

$$\frac{\partial N}{\partial x} = y^3 - 4$$

$$\therefore \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$$

$$= \frac{-3\cdot\left(y^3 + 2\right)}{y\left(y^3 + 2\right)}$$

$$= -\frac{3}{y} = \text{ function of y alone}$$

$$\therefore \text{ I.F. } = e^{\int f(y)dy}$$

$$= e^{-3\cdot\int \frac{1}{y}dy}$$

$$= e^{-3 \log y}$$

$$= e^{\log\left(\frac{1}{y^3}\right)}$$

$$I.F. = \frac{1}{y^3}$$

$$\therefore \frac{\left(y^4 + 2y\right)}{y^3} dx + \frac{\left(xy^3 + 2y^4 - 4x\right)}{y^3} \cdot dy = 0$$

This is exact differential equation

Comparing with Mdx+ Ndy=0; we get

$$M = y + \frac{2}{y^2}$$

$$N = x + 2y - 4\frac{x}{y^3}$$

General solution is given by

$$\int M\,(treat\ \text{y constant})\,\text{dx} + \int N\ (\text{terms free from x})\,\text{dy} = \text{c}$$

$$\therefore \int \left(y + \frac{2}{y^2}\right) dx + 2 \cdot \int y\,dy = c$$

$$\therefore \left(y + \frac{2}{y^2}\right) \int dx + 2\frac{y^2}{2} = c$$

$$\left(y + \frac{2}{y^2}\right) x + y^2 = c$$

This is required general solution.

Check your progress:

Solve :

i) $\left(2xy^4e^y + 2xy^3 + y\right) dx + \left(x^2y^4e^y - x^2y^2 - 3x\right) dy = 0$

Hint : I.F. $= \frac{1}{y^4}$..........

General solution is given by

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

ii) $x^2y^3dx + \left(x^3y - 2\right) dy = 0$

Ans $3x^3y - 2y - 6 = cy \cdot e^{3/y}$

8.4 LINEAR EQUATION AND EQUATIONS REDUCIBLE TO LINEAR FORM

The first order and first degree linear -

Differential equation is of the type

$$\frac{dy}{dx} + py = Q$$

Where y is dependent variable and x is independent variable. and p& Q are functions of x only. (may be constant)

The above differential equation is known as Leibnitz‟s linear differential equation.

Working Rule:

1) Consider linear differential equation.

$$\frac{dy}{dx} + py = Q$$

Where P and Q are function of x or constants only

Its integrating factor is given by

$$I.F. = e^{\int pdx}$$

Its solution is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (I.F.)dx + c$$

Where c is arbitrary constant.

2) For linear differential equation

$$\frac{dx}{dy} + p_1 x = Q_1$$

Where p_1 and Q_1 are functions of y or constants only

Its integrating factor is given by

$$\therefore \text{ I.F.} = e^{\int p_1 dy}$$

Its solution is given by

$$x \cdot (IF) = \int Q\ (IF)\ dy + c$$

Where c is arbitrary constant.

Solved Examples:-

Example 4: Solve $(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$

Solution: The given equation is

$$(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$$

Dividing throughout by $(x+1)$ we have

$$\therefore \frac{dy}{dx} - \frac{1}{(x+1)} \cdot y = e^x (x+1) \quad \text{.........(1)}$$

This is of the type

$$\therefore \frac{dy}{dx} + py = Q$$

Hence equation (1) is linear differential equation.

Where

$$P = -\frac{1}{(x+1)},\ Q = e^{x}(x+1)$$

$$\therefore\ \text{I.F.} = e^{\int \text{pdx}}$$

$$= e^{-\int \frac{1}{x+1}dx}$$

$$= e^{-\log(x+1)}$$

$$I.F. = e^{\log\left(\frac{1}{x+1}\right)}$$

$$I.F. = \frac{1}{x+1}$$

Hence the solution of differential equation (1) is

$$y\cdot(\text{I.F.}) = \int Q\ (\text{IF})\,dx + c$$

$$\therefore\ y\cdot\frac{1}{x+1} = \int e^{x}(x+1)\ \frac{1}{(x+1)}\,dx + c$$

$$\therefore\ \frac{y}{x+1} = \int e^{x}\cdot dx + c$$

$$\therefore\ \frac{y}{x+1} = e^{x} + c$$

$$\therefore\ y = (e^{x} + c)\cdot(x+1)$$

This is the required solution.

Example 5: Solve $(1+y^{2})\,dx = (\tan y^{-1} - x)\,dy$

Solution: The given equation is

$$\therefore\ (1+y^{2})\,dx = (\tan y\ - x)\,dy$$

$$\therefore\ \frac{dx}{dy} = \frac{\tan y^{-1} - x}{1+y^{2}}$$

$$\therefore\ \frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^{2}} - \frac{1}{1+y^{2}}\cdot x$$

$$\therefore\ \frac{dx}{dy} + \frac{1}{1+y^{2}}\cdot x = \frac{\tan^{-1} y}{1+y^{2}}\ ....(1)$$

This is of the type

$$\frac{dx}{dy} + px = Q$$

Where $p = \frac{1}{1+y^{2}},\ Q = \frac{\tan^{-1} y}{1+y^{2}}$

Hence equation (i) is a linear differential equation

$$\therefore\ \text{I.f} = e^{\int \text{pdy}}$$

$$= e^{\int \frac{1}{1+y^{2}}\cdot dy}$$

$$I.F. = e^{\tan^{-1} y}$$

The solution of differential equation (i) is

$$x(I.F.) = \int Q\ (\text{I.F.})dy + c$$

$$\therefore\ x \cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1}y} \cdot dy + c$$

consider the integral

$$\int \frac{\tan^{-1}y}{1+y^2}\ e^{\tan^{-1}y} \cdot dy$$

put $z = \tan^{-1}y$

Differentiating with respect to z

$$1 = \frac{1}{1+Y^2} \cdot \frac{dy}{dz}$$

$$\therefore\ \frac{1}{1+Y^2} \cdot dy = dz$$

$$\therefore\ \int z \cdot e^z \cdot dz$$

$$= z \cdot \int e^z \cdot dz - \left(\int \frac{d}{dz} z \int e^z \cdot dz \right) dz$$

$$= z \cdot e^z - \int l \cdot e^z \cdot dz$$

$$= z \cdot e^z - e^z$$

$$= e^z (z-1)$$

put $z = \tan^{-1}y$

$$= e^{\tan^{-1}y} (\tan^{-1} y - 1)$$

$\therefore$ solution is given by

$$x \cdot e^{\tan^{-1}y} = e^{\tan^{-1}y} (\tan^{-1} y - 1) + c$$

$$\therefore\ x = \tan^{-1}y - 1 + c \cdot e^{-\tan^{-1}y}$$

This is the required solution.

Example 6: Solve

$$x(1-x^2)\frac{dy}{dx} + (2x^2-1)y = x^3$$

Solution: The given equation is

$$x(1-x^2)\frac{dy}{dx} + (2x^2-1)y = x^3$$

$\div$ through out by $x(1-x^2)$ we have

$$\therefore\ \frac{dy}{dx} + \frac{(2x^2-1)}{x(1-x^2)} y = \frac{x^3}{x\ (1-x^2)} \quad \text{..........(1)}$$

Hence equation (1) is linear in dependent variable y This is of the type

$$\frac{dy}{dx} + py = Q$$

$$where\ P = \frac{(2x^2 - 1)}{x(1-x^2)}, Q = \frac{x^3}{x(1-x^2)}$$

$$\therefore\ I.F. = e^{\int pdx}$$

$$Let\ P = \frac{2x^2 - 1}{x(1-x)(1+x)}$$

$$P = -\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$

(*By partial* fraction)

$$\therefore\ IF = e^{\int\left[-\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}\right]\cdot dx}$$

$$= e^{-\log x + \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)}$$

$$= e^{-\left[\log x\cdot\sqrt{1-x^2}\right]}$$

$$= e^{\log\cdot\left[x\cdot\sqrt{1-x^2}\right]^{-1}}$$

$$IF = \frac{1}{x\sqrt{1-x^2}}$$

Hence solution of differential equation (i) is

$$y\ (IF) = \int Q(IF)dx + c$$

$$\therefore\ y\cdot\frac{1}{x\sqrt{1-x^2}} = \int\frac{x^2}{(1-x^2)}\cdot\frac{1}{x\sqrt{1-x^2}}\cdot dx + c$$

$$= \int\frac{x}{(1-x^2)^{3/2}}\cdot dx + c$$

$$= -\frac{1}{2}\cdot\int(-2x)(1-x^2)^{3/2}\cdot dx + c$$

$$= -\frac{1}{2}\left[\frac{(1-x^2)^{-1/2}}{-1/2}\right] + c$$

$$\left\{\int f^n\cdot f^1 = \frac{f^{n+1}}{n+1}\right.$$

$$\therefore\ \frac{y}{x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + c$$

$$\therefore\ y = x + cx\sqrt{1-x^2}$$

Which is the required solution.

Example 7: Solve

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\cdot\frac{dx}{dy}=1$$

Solution: The given equation is

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\cdot\frac{dx}{dy}=1$$

$$\therefore\ \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\ldots\ldots..(1)$$

Which is of the type

$$\frac{dy}{dx}+py=Q$$

where $P=\frac{1}{\sqrt{x}}, Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

The equation (1) is linear in y

$\therefore$ I.F. $=e^{\int pdx}$

$=e^{\int\frac{1}{\sqrt{x}}\cdot dx}$

$I.F.=e^{2\sqrt{x}}$

Hence the solution of differential equation (1) is

$$y\cdot(IF)=\int Q\cdot(\text{IF})dx+c$$

$$y\cdot e^{2\sqrt{x}}=\int\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\cdot e^{2\sqrt{x}}\cdot dx+c$$

$$=\int\frac{1}{\sqrt{x}}dx+c$$

$$y\cdot e^{2\sqrt{x}}=2\sqrt{x}+c$$

This is the required general solution.

Example 8: Solve $\left(1+y^2\right)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}$

Solution: The given equation is

$$\left(1+y^2\right)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}$$

$$\therefore\ \left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=-\left(1+y^2\right)$$

$$\therefore\ x-e^{\tan^{-1}y}=-\left(1+y^2\right)\cdot\frac{dx}{dy}$$

$$\therefore\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{\tan^{-1}y}}{1+y^2}\ldots\ldots..(1)$$

Which is of the type

$$\frac{dx}{dy}+px=Q$$

where $p = \frac{1}{1+y^2}, Q = \frac{e^{\tan^{-1} y}}{1+y^2}$

The equation (1) is linear differential equation

Hence

$$IF = e^{\int p dy}$$

$$= e^{\int \frac{1}{1+y^2} \cdot dy}$$

$$IF = e^{\tan^{-1} y}$$

Hence solution of differential equation (1) is given by

$$x \cdot (IF) = \int Q \text{ (IF) } dy + c$$

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y} \cdot dy + c$$

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{2\tan^{-1} y}}{1+y^2} \cdot dy + c ....(2)$$

put $\tan^{-1} y = t$

$$\therefore \frac{1}{1+y^2} \cdot dy = dt$$

$\therefore$ equation (2) becomes

$$x \cdot e^{\tan^{-1} y} = \int e^{2t} \cdot dt + c$$

$$x \cdot e^{\tan^{-1} y} = \frac{e^{2t}}{2} + c$$

put $t = \tan^{-1} y$

$$\therefore x \cdot e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + c$$

$$\therefore 2x \cdot e^{\tan^{-1} y} = e^{2\tan^{-1} y} + c_1 \text{ where } c_1 = 2c$$

This is the required general solution.

Check your progress:

1) Solve

i) $(2y + x^2)dx = xdy$

Ans: $y = x^2 \log(cx)$

ii) $\frac{dy}{dx} + \frac{y}{1-x} = x^2 - x$

Ans : $2y = (1-x)(c^2 - x^2)$

iii) $(x^2 + 1) \cdot \frac{dy}{dx} = x^3 - 2xy + x$

Ans : $(x^2 + 1)y = \frac{x^4}{4} + \frac{x^2}{2} + c$

iv) $\frac{dy}{dx} + \frac{x}{\left(1-x^2\right)^{3/2}} \quad y = \frac{x\left(1+\sqrt{1-x^2}\right)}{\left(1-x^2\right)^2}$

$H\text{int I.F.} = e^{-\frac{1}{\sqrt{1-x^2}}}$

$y = \frac{1}{\sqrt{1-x^2}} + c \cdot c^{\frac{1}{\sqrt{1-x^2}}}$

v) $dx + xdy = e^{-y}\sec^2 \cdot dy$

$H\text{int : I.F.} = e^y$

$x \cdot e^y = \tan y + c$

vi $x\cos x \cdot \frac{dy}{dx} + (\cos x - x\sin x) \cdot y = 1$

$H\text{int : I.F.} = \frac{x}{\sec x}$

$xy\cos x = x + c$

vii $\left(x^2+1\right)^3 \cdot \frac{dy}{dx} + 4x \cdot \left(x^2+1\right)^2 \cdot y = 1$

$H\text{int : If} = \left(x^2+1\right)^2$

$\left(x^2+1\right)^2 \cdot y = \tan^{-1} x + c$

viii $(x+y+1) \cdot \frac{dy}{dx} = 1$

$H\text{int : I.F.} = e^{-y}$

$x + y + 2 = c \cdot e^y$

ix $\left(x+2y^3\right) \cdot dy = ydx$

$H\text{int I.F.} = \frac{1}{y}$

$x = y^3 + cy$

8.5 EQUATIONS REDUCIBLE TO LINEAR FORM

I) Bernoulli‟s Equation :

The equation of the form

$$\frac{dy}{dx} + py = Q \cdot y^n$$

is called as Bernoulli's equations

$\div$ throughout by y^n, *we get*

$\therefore\ y^{-n} \cdot \frac{dy}{dx} + P \cdot y^{1-n} = Q$.........(1)

Let $y^{1-n} = u$

$$\therefore\ (1-n)\cdot y^{-n}\cdot\frac{dy}{dx} = \frac{du}{dx}$$

using equation (1) we get

$$\therefore\ \frac{1}{1-n}\cdot\frac{du}{dx} + Pu = Q$$

$$\therefore\ \frac{du}{dx} + (1-n)\cdot pu = (1-n)\ Q$$

This is Bernoulli‟s differential equation and can be solved.

Note: The equation is also Bernoulli‟s equation

We divide by x^n and substitute u= x^{1-n} and proceed.

Solved Examples:-

Example 9: Solve $\frac{dy}{dx} + \frac{y}{x} = xe^x\cdot y^2$

Solution:

$$\frac{dy}{dx} + \frac{y}{x} = xe^x\cdot y^2 ....(1)$$

Which is of the type

$$\frac{dy}{dx} + Py = Q\ \cdot y^n ....$$

Where $p = \frac{1}{x}$, $Q = xe^x, n = 2$

Equation (1) *is* Bernoulli's differential equation

$\div$ throughtout by y^2, *we get*

$$\therefore\ y^{-2}\cdot\frac{dy}{dx} + \frac{1}{x}\cdot y^{-1} = x\cdot e^x ....(2)$$

Put $y^{-1} = u$

Differentiating with respect to x

$$\therefore\ -1\cdot y^{-2}\cdot\frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore\ y^{-2}\cdot\frac{dy}{dx} = -\frac{du}{dx}$$

using equation 2 we get

$$\therefore\ -\frac{du}{dx} + \frac{1}{x}\cdot u = x\cdot e^x$$

$$\therefore\ \frac{du}{dx} - \frac{1}{x}\cdot u = x\cdot e^x$$

Which is linear differential equation.

where $p = \frac{1}{x}, Q = -x\ e^x$

$\therefore I.F. = e^{\int Pdx}$

$= e^{-\int \frac{1}{x} dx}$

$= e^{-\log x}$

$I.F. = e^{\log\left(\frac{1}{x}\right)}$

$\therefore$ I.F. $= \frac{1}{x}$

Hence, General solution is given by

$$u \cdot (IF) = \int Q \, (\text{IF}) \, dx + c$$

$$\therefore \; u \cdot \frac{1}{x} = \int -x \, e^x \, \frac{1}{x} \, dx + c$$

Put $u = y^{-1}$

$\therefore \; y^{-1} \cdot \frac{1}{x} = -\int e^x dx + c$

$\frac{1}{xy} = -e^x + c$

This is the required solution.

Example 10: Solve $xy\left(1 + xy^2\right) \cdot \frac{dy}{dx} = 1$

Solution: The given equation is

$$xy \cdot \left(1 + xy^2\right) \cdot \frac{dy}{dx} = 1$$

$$\therefore \; \frac{dx}{dy} = xy + x^2 y^3$$

$$\therefore \; \frac{dx}{dy} - xy = x^2 y^3 \text{.........(1)}$$

which is of the type,

$$\frac{dx}{dy} + px = Q \cdot x^n$$

where $p = -y, \; Q = y^3, \; n = 2$

Equation 1 is a Bernoulli‟s differential equation

$\div$ through out by x^2, *we get*

$\therefore \; x^{-2} \cdot \frac{dx}{dy} - x^{-1} \cdot y = y^3$.........(2)

Let $x^{-1} = u$

$$\therefore \; -x^{-2} \cdot \frac{dx}{1y} = \frac{du}{dy}$$

$\therefore \; x^2 \cdot \frac{dx}{dy} = -\frac{du}{dy}$

Thank You

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Just another Way To Learn