Applied Mathematics

Part-2

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Just another Way To Learn

So we proceed as follows

Expanding by R_1

$$x_1 + x_2 + x_3 = 0$$

Assume any element to be zero say x_1 and give any conventional value say 1 to x_2 and find x_3

Let

$x_1 = 0, \ \ x_2 = 1$

$\therefore \ x_3 = -1$

$$\therefore \ x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Case (iii) :- Let x=1

Again consider

$x_1 + x_2 + x_3 = 0$

Let $x_2 = 0, \ \ x_1 = 1$

$\therefore \ x_3 = -1$

$$\therefore \ x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Type (iii) :- A is symmetric and eigen values are repeated

Example 6: Find eigen values and eigen vectors for .

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution:

Step :- Characteristic equations of A in a is

$$[A - a\ I] = 0$$

$$\begin{bmatrix} 6-a & -2 & 2 \\ -2 & 3-a & -1 \\ 2 & -1 & 3-a \end{bmatrix} = 0$$

$[A] = 32$

i.e. $a^3 - 12a^2 + (8+14+14)\ a - 32 = 0$

$\therefore \ \ a^3 - 12a^2 + 36\ a - 32 = 0$

$(a - 2)$ is a factor

Synthetic division :-

2	1	−12	36	−32
		2	−20	32
	1	−10	16	0

$$a^2 - 10a + 6$$
$$= (a-8)(a-2)$$
$$\therefore\ a^3 - 12a^2 + 36a - 32 = 0$$
$$(a-2)(a-2)(a-8) = 0$$

$\therefore\ a = 8,\ 2,\ 2$

Step (ii) :- Matrix equation is

$$\begin{bmatrix} 6-a & -2 & 2 \\ -2 & 3-a & -1 \\ 2 & -1 & 3-a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i) :- For $a = 8$

Matrix equation is given by

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ \frac{x_1}{12} = \frac{-x_2}{6} = \frac{x_3}{6}$ ....*By* cramer's rule

$$\frac{x_1}{2} = \frac{-x_2}{-1} = \frac{x_3}{1}$$

$$\therefore\ x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case (ii) :- Let $a = 2$

Matrix equation is given by

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding R_1

$4x_1 - 2x_2 + 2x_3 = 0$

Let $x_1 = 0,\ \ x_2 = 1$

$\therefore\ \ x_3 = 1$

$$\therefore\ x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii) :- Let

$a = 2$

$\therefore$ A is symetric

$\therefore\ x_1,\ x_2,\ x_3$ are orthogonal

$$\text{Let, } x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$\because$ x_1, x_3 are orthogonal

$\therefore$ $x_1^1, x_3 = 0$

$\therefore$ $2l - m + n = 0$..........(1)

x_2, x_3 are orthogonal

$\therefore$ $x_2^1, x_3 = 0$

$\therefore$ $ol + m + n = 0$..........(2)

solving (1) and (2) by cramer's rule

$$\frac{l}{-2} = \frac{-m}{2} = \frac{n}{2}$$

$$\therefore \frac{l}{+1} = \frac{m}{1} = \frac{n}{-1}$$

$$\therefore x_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Check your progress:

1) Find eigen values and eigen vectors for

i) $$A = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans :- Eigen values are 0,1,2

$$\therefore x = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Ans *Eigen values* are 2,3 and 6

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(iii) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ 1 & -2 & 0 \end{bmatrix}$

Ans : *Eigen values* are 5, -3,-3

$x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $x_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

3.6 SUMMARY

In this chapter we have learn

- ❖ Linearly dependent & independent vector.
- ❖ Inner product of two vector i.e same as dot product 7 its properties.
- ❖ Characteristics equation & its root by using

$|A - ª I| = 0$

❖Eigen vector which is corresponding to Eigen value which we get from $|A - ª I| = 0$

3.7 UNIT END EXERCISE

1) Is the system of vector $x_1 = (2,2,1)^T, x_2 = (1,3,1)^T$ linear by dependent?

2) Show that the vectors (1,2,3) (2,20) form a linearly independent set.

3) Show that the following vector are linearly dependent & find the relation between them

$x_1 = (1,-1,1), x_2 = (2,1,1), x_3 = (3,0,2)$

4) Prove the properties of an inner product.

i. $< X, Y >= 3x_1y_1 + 4x_2y_2$.

ii. $< X, Y >= 9x_1y_1 - 3x_2y_2 + 4x_3y_3$

5) Find Eigen value and Eigen vector for the following matrix.

i) $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$

iv) $A = \begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$

v) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

vi) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

vii) $A = \begin{bmatrix} Cos0 & -Sin\_ \\ Sin\_ & Cos0 \end{bmatrix}$

viii) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

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4

CAYLEY – HAMILTON THEORY

UNIT STRUCTURE

4.1 OBJECTIVE

After going through this chapter you will able to

- Find by using Cayley Hamilton Theorem.
- Application of Cayley- Hamilton Theorem.
- Find diagonal matrix on similar matrix.
- Characteristic Polynomial & Minimal Polynomial of matrix A.
- Derogatory & non-derogatory matrix.
- Complex matrix like Hermitian, Skew-Hermitian unitary matrix.

2. INTRODUCTION

In previous chapter we learn about Eigen values & Eigen Vector. How here we are going to discuss Cayley Hamilton Theory & its application also we had study only Real matrix. We introduce here complex matrix with type of complex matrix also minimal polynomial.

3. CAYLEY – HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation. If the characteristic Equation for the n^{th} order square matrix A is $|A - aI| = (-1)^n\left[a^n + a_1 a^{n-1} + a_2 a^{n-2} .... + a_n\right]$ then

$$(-1)^n\left(A^n + a_1 A^{n-1} + a_2 A^{n-2} .... + a_n I\right) = 0$$

Example 1:

Show that the given matrix A satisfies its characteristic equation.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

The characteristic equation of the matrix A is $|A - aI| = 0$

$$\begin{vmatrix} 2-a & 1 & 1 \\ 0 & 1-a & 0 \\ 1 & 1 & 2-a \end{vmatrix} = 0$$

$$\therefore (2-a)\left[(1-a)(2-a)-0\right]-1(0)+1\left(0-(1-a)\right)=0$$

$$\therefore (2-a)\left[2-3a+a^2\right]+1(-1+a)=0$$

$$\therefore 4-6a+2a^2-2a+3a^2-a^3-1+a=0$$

$$\therefore -a^3+5a^2-7a+3=0$$

$$\therefore a^3-5a^2+7a-3=0$$

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 7A - 3I = 0$ ………………………..(1)

Now, we have

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$\therefore A^3 - 5A^2 + 7A - 3I =$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} -5 \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} +7 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} -3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix}14 & 13 & 13\\ 0 & 1 & 0\\ 13 & 13 & 14\end{bmatrix} -5 \begin{bmatrix}5 & 4 & 4\\ 0 & 1 & 0\\ 4 & 4 & 5\end{bmatrix} +7\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix} -3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}14 & 13 & 13\\ 0 & 1 & 0\\ 13 & 13 & 14\end{bmatrix} - \begin{bmatrix}25 & 20 & 20\\ 0 & 5 & 0\\ 20 & 20 & 25\end{bmatrix} + \begin{bmatrix}14 & 7 & 7\\ 0 & 7 & 0\\ 7 & 7 & 14\end{bmatrix} - \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}28 & 20 & 20\\ 0 & 8 & 0\\ 20 & 20 & 28\end{bmatrix} - \begin{bmatrix}28 & 20 & 20\\ 0 & 8 & 0\\ 20 & 20 & 28\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$=0$$

$$\therefore A^3 - 5A^2 + 7A - 3I = 0$$

Thus the matrix A satisfies its characteristic equation.

Example 2 :

Calculate A^7 by using Cayley Hamilton theorem.

Where $A = \begin{bmatrix}3 & 6\\ 1 & 2\end{bmatrix}$

Solution :

The characteristic equation of A is

$$|A - a I| = 0$$

$$\begin{bmatrix}3 - a & 6\\ 1 & 2 - a\end{bmatrix} = 0$$

$$(3 - a)(2 - a) - 6 = 0$$

$$6 - 2a - 3a + a^2 - 6 = 0$$

$$\therefore a^2 - 5a = 0$$

By Cayley Hamilton theorem,

$$A^2 - 5A = 0$$

i.e. $A^2 = 5A$

Now to calculate

$A^7 = A^5.A^2 = A^5.5A = 5A^6$

$= 5A^4.A^2 = 25A^5$

$= 25A^3.A^2 = 125A^4$

$= 125A^2.A^2 = 125(5A).(5A)$

$= 3125A^2 = 3125(5A)$

$= 15625A$

$$A^7 = 15625\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 46875 & 93750 \\ 15625 & 31250 \end{bmatrix}$$

$\therefore$ The value of $A^7 = \begin{bmatrix} 46875 & 93750 \\ 15625 & 31250 \end{bmatrix}$

Example 3:

By using Cayley Hamilton theorem find A^{-1}

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

The characteristics equation of A is

$|A - \lambda I = 0|$

$$\begin{bmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$$

$(1-\lambda)[1-2\lambda+\lambda^2-4]+1[\lambda-1-2]+1[-2+\lambda-1] = 0$

$\lambda^2 - 2\lambda - 3 + 3\lambda + 2\lambda^2 - \lambda^3 + \lambda - 3 - 3 + \lambda = 0$

$-\lambda^3 + 3\lambda^2 + 3\lambda - 9 = 0$

$\lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0$

By Cayley Hamilton theorem

$A^3 - 3A^2 - 3A + 9I = 0$

Multiply by A^{-1}

$$\therefore\ A^3A^{-1}-3A^2A^{-1}-3AA^{-1}+9IA^{-1}=0A^{-1}$$

$$\therefore\ A^2-3A-3I+9A^{-1}=0$$

$$A^{-1}=\frac{1}{9}\left[3A+3I-A^2\right] \qquad \text{.......(1)}$$

$$A^2=\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}=\begin{bmatrix}3 & 0 & 0\\ 0 & 6 & 3\\ 0 & 3 & 6\end{bmatrix}$$

$$3A+3I-A=3\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}+3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}3 & 0 & 0\\ 0 & 6 & 3\\ 0 & 3 & 6\end{bmatrix}$$

$$3A+3I-A^2=\begin{bmatrix}3 & -3 & 3\\ -3 & 3 & 6\\ 3 & 6 & 3\end{bmatrix}+\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}-\begin{bmatrix}3 & 0 & 0\\ 0 & 6 & 3\\ 0 & 3 & 6\end{bmatrix}$$

$$=\begin{bmatrix}3 & -3 & 3\\ -3 & 0 & 3\\ 3 & 3 & 0\end{bmatrix}$$

$$A^{-1}=\frac{1}{9}\left[3A+3I-A^2\right]$$

$$=\frac{1}{9}\begin{bmatrix}3 & -3 & 3\\ -3 & 0 & 3\\ 3 & 3 & 0\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}1 & -1 & 1\\ -1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$$

$$A^{-1}=\frac{1}{3}\begin{bmatrix}1 & -1 & 1\\ -1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$$

Check your progress:

1) Find the characteristic polynomial of the matrix.

$A=\begin{bmatrix}3 & 1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$ Verify Cayley-Hamilton theorem for this matrix.

Hence find A^{-1}

Ans: $A^{-1} = \frac{1}{20}\begin{bmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{bmatrix}$

2) Use Cayley-Hamilton theorem to find inverse of the matrix.

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$ Ans: $\frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$

3) Use Cayley-Hamilton theorem to find the inverse of

$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$ Ans: $A^{-1} = \frac{1}{7}\begin{bmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{bmatrix}$

4) Show that the following matrices satisfy their characteristics equation

$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

5) Using the characteristics equation show that inverse of the matrix

i) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

ii) $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

iii) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Ans: $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

4.4 SIMILARITY OF MATRIX

Two matrix A and B of order nxn over F are said to be similar if there exist a non-singular matrix P (invertible matrix) of order nxn such that $B = P^{-1}AP$

This transformation of matrix A by a non-singular matrix P to B is called a similarity transformation.

Diagonal matrix: If a square matrix A of order n has linearly independent eigen vectors then matrix P can be formed such that $P^{-1}AP$ is diagonal matrix i.e.

$D = P^{-1}AP$

Example 4:

Two similar matrices A and B have the same eign values.

Solutions:

Since A and B are similar, there exists a non-singular matrix P such that

$B = P^{-1}AP$

Consider $\quad |B - \lambda I| = |P^{-1}AP - \lambda I|$

$$|B - \lambda I| = |P^{-1}AP - \lambda P^{-1}IP|$$

$$= |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}||A - \lambda I||P|$$

$$= |A - \lambda I||P^{-1}||P|$$

$$= |A - \lambda I| \qquad \because \ |P^{-1}|.|P| = 1$$

$$\therefore |B - \lambda I| = |A - \lambda I|$$

Hence the characteristics equation of A and B are the same

$\therefore$ A and B have same eigen values.

Example 5:

Show that $A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ have same characteristics equations but A and B not similar matrices.

Solutions:

Let $A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

Characteristics equation of A is $|A - aI| = 0$

i.e. $\therefore \begin{vmatrix} 1-a & 1 \\ 0 & 1-a \end{vmatrix} = (1-a)^2 = a^2 - 2a + 1 = 0$ s equation

$\therefore$ Characteristics equation of B is

$|B - aI| = 0$

i.e. $\begin{vmatrix} 1-a & 0 \\ 0 & 1-a \end{vmatrix} = (1-a)^2 = a^2 - 2a + 1 = 0$

$\therefore$ Characteristics equation of A = Characteristics equation of B

Now we will show that A and B are not similar

Suppose $A \quad B$

$\therefore$ There exist non-singular matrix C such that, $B \ = 0 \ C^{-1}AC$

Let $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = 2, \ \therefore$ C is non-singular as $= [C] \neq 0$

$\therefore C^{-1}$ exists

adj $C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$C^{-1} = \frac{1}{[C]} \text{ adj } (C) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$C^{-1}AC = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq B$$

Hence A and B are not similar matrices.

Example 6: Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, Find similarity to a diagonal matrix.

Find the diagonal matrix.

Ans : $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

4.5 CHARACTERISTICS POLYNOMIAL

Solving the determinant $[A-\lambda I]$, a polynomial is obtained which is called as a characteristics polynomial.

For e.g. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristics polynomial is given by

$$|\propto||A-\lambda I| = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

$$=(2-\lambda)\left[(2-\lambda)^2-1\right]+1\left[-(2-\lambda)+1\right]+1\left[1-(2-\lambda)\right]$$

$$=(2-\lambda)\left[\lambda^2-4\lambda+3\right]+2\lambda-2-\lambda^3+6\lambda^{2}-3\lambda+4$$

$$=\lambda^3-6\lambda^2+9\lambda-4$$

4.6 MINIMAL POLYNOMIAL

Monic Polynomial: A Polynomial in λ, in which the coefficient of the highest power of λ is unity is called a monic polynomial.

For e.g. $\lambda^5+2\lambda^4+3\lambda^3-6\lambda+5$ is a monic polynomial of degree polynomial.

If a polynomial f annihilates A then ~ f also f annihilates. A for $\sim \in R$, therefore there exists a monic polynomial annihilating A.

If the characteristics roots of the characteristics equation are distinct then f (λ) = 0 is called minimal equation.

If matrix of order 3x3 are having characteristics root 2,3,3 then,

$$(\lambda-2)(\lambda-3)=0$$

Or $(A-2)(A-3)=0$ is the minimal equation.

Hence the degree of the equation is 2 and less than the order of the polynomial.

Derogatory Matrix: A nxn matrix is called derogatory if the degree of its minimal polynomial is less than n.

Non-Derogatory Matrix: A nxn matrix is called non-derogatory if the degree of minimal polynomial is equal to n.

Properties of Minimal Polynomial:

(1) There exists a unique minimal polynomial of the matrix A.

(2) The minimal polynomial of A divides the characteristics polynomial of A.

(3) If λ is the root of the minimal polynomial of A then λ is also characteristics of root of A.

(4) If the n characteristics of root of A are distinct then A is non derogatory.

Example 7:

Check whether the following matrix is derogatory or non derogatory also find its minimal polynomial.

i) $$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution:

The characteristics polynomials of matrix A is

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}$$

$$= \lambda^3 - (sum\ of\ diagonal\ element\ of\ A)\lambda^2 +$$
$$(sum\ of\ minor\ of\ diagonal\ element\ of\ A)\lambda - |A|$$

$$= \lambda^3 - [2+1-1]\lambda^2 + \left[\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}\right]\lambda - \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$$\therefore \lambda^3 - 2\lambda^2 + [4-4-5]\lambda - (-6)$$

$\therefore a^3 - 2a^2 - 5a + 6$

$\therefore (a+2)(a-1)(a-3)$

$\therefore$ The characteristics roots are -2, 1 and 3 which are distinct.

Therefore matrix A is non-derogatory.

ii) $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution:

The characteristics polynomials of matrix A is

$$|A - aI| = \begin{bmatrix} 2-a & 1 & 1 \\ 2 & 3-a & 2 \\ 3 & 3 & 4-a \end{bmatrix}$$

$= a^3 - (sum\ of\ diagonal\ element\ of\ A)a^2 +$
$(sum\ of\ minor\ of\ diagonal\ element\ of\ A)a - |A|$

$$= a^{3-}[2+3+4]a^2 + \left[\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}\right] - \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix}$$

$= a^3 - 9a^2 + [6+5+4]a - 7$

$= a^3 - 9a^2 + 15a - 7$

$= (a-1)(a-1)(a-7)$

$\therefore$ The characteristics roots are 1, 1 and 7 which are not distinct.

Therefore matrix A is derogatory.

Example 8:

Show that the matrix A is derogatory also find its minimal polynomial.

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

Solution:

The characteristics polynomials of matrix A is

$$|A-\lambda I| = \begin{bmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{bmatrix}$$

$= \lambda^3 - (sum\ of\ diagonal\ element\ of\ A)\lambda^2 +$

$(sum\ of\ minor\ of\ diagonal\ of\ matrix\ A)\lambda - |A|$

$$= \lambda^3 - [1+4-3]\lambda^2 + \left[\begin{vmatrix} 4 & 2 \\ -6 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -4 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -6 \\ 0 & 4 \end{vmatrix}\right]\lambda - \begin{vmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{vmatrix}$$

$= \lambda^3 - 2\lambda^2 + [0-3+4]\lambda - 0$

$= \lambda^3 - 2\lambda^2 + \lambda$

$= \lambda(\lambda^2 - 2\lambda + 1)$

$= \lambda(\lambda-1)(\lambda-1)$

$\therefore$ The characteristics roots are 0, 1 & 1 which are not distinct.

Therefore matrix A is derogatory matrix.

But we know that characteristic root of A is root of minimal polynomial.

$\therefore f(\lambda) = \lambda(\lambda-1) = \lambda^2 - \lambda.$

Now check whether $= f(\lambda)$.annihilated matrix A.

$\therefore f(\lambda) = A^2 = A$

$$A^2 = A.A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

$$A^2 - A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} - \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

$A^2 - A = 0$

$\therefore f(A) = 0$

$\therefore$ The minimal of polynomial of A is $f(\lambda) = \lambda^2 - \lambda$

& degree of polynomial is 2 which is less than 3
Hence matrix A is derogatory.

Example 9:

Find the minimal polynomial and show that it is derogatory matrix.

Where, $A=\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution:

The characteristics polynomials of matrix A is

$$|A-\lambda I|=\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$=(2-\lambda)\left[(3-\lambda)(2-\lambda)-2\right]-2\left[2-\lambda-1\right]+1(2-3+\lambda)$$

$$=(2-\lambda)\left[\lambda^2-5\lambda+6-2\right]-2\left[-\lambda+1\right]+\lambda-1$$

$$=-\lambda^3+5\lambda^2-4\lambda+2\lambda^2-10\lambda+8-3\lambda-3$$

$$=-\lambda^3+7\lambda^2-11\lambda+5$$

$$=(\lambda-1)(\lambda-1)(\lambda-5)$$

$\therefore$ The characteristics roots of matrix A are 1, 1 and 5.

$\because$ roots are.

$\therefore$ The matrix A is derogatory.

But we know that characteristics root of A is also a root of its minimal polynomial.

$$=\therefore f(\lambda)=(\lambda-1)(\lambda-5)=\lambda^2-6\lambda+5$$

Now check whether $f(\lambda)$ annihilated matrix A i.e.

$$f(A)=A^2-6A+5I=0..............(I)$$

$$A^2-6A+5I=\begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix}-6\begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}+5\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} - \begin{vmatrix} 12 & 12 & 6 \\ 6 & 18 & 6 \\ 6 & 12 & 12 \end{vmatrix} + \begin{vmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{vmatrix}$$

$$\therefore \ f(A) = 0$$

$\therefore$ The minimal of polynomial of A is $f(a) = a^2 - 6a + 5$

And degree of polynomial is 2 which is less than 3

$\therefore$ The matrix A is derogatory.

Check Your Progress:

(1) Show that the following matrices are derogatory and hence find the minimal polynomial.

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ Ans: $a^2 - 3a + 2 = 0$

(ii) $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$ Ans: $a^2 - a = 0$

(2) Check whether the following matrix is derogatory or non-derogatory also find the minimal polynomial.

(i) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ Ans: Non – derogatory

(ii) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Ans: Derogatory

4.7 COMPLEX MATRICES

Z =x+iy is called a complex number, where $i=\sqrt{-1}$ and $x, y \in R$ and $\bar{Z}=x-iy$ is called a conjugate of the complex number Z

Let A be a mxn matrix having complex numbers as its elements, then the matrix is called a complex matrix.

Conjugate of a Matrix:

The matrix of order mxn is obtained by replacing the elements by their corresponding conjugate elements, is called conjugate of a matrix. It is denoted by $\bar{A}$

For e.g. $A=\begin{vmatrix} 2-3i & 1-i & 3 \\ 2i+1 & 2 & 2i-3 \end{vmatrix}$

$\bar{A}=\begin{vmatrix} 2+3i & 1+i & 3 \\ -2i+1 & 2 & -2i-3 \end{vmatrix}$

Properties of conjugate of matrix:

(1) $\overline{(\bar{A})}=A$

(2) $\overline{A+B}=\bar{A}+\bar{B}$

(3) $\overline{(AB)}=\bar{A}.\bar{B}$

Conjugate Transpose:

Transpose of the conjugate matrix A is called conjugate transpose. It is denoted by A^{-} .

For e.g. $A=\begin{vmatrix} 1+i & -i & 1 \\ 3 & i+2 & 3i-2 \end{vmatrix}$

$\bar{A}=\begin{vmatrix} 1-i & i & 1 \\ 3 & -i+2 & -3i-2 \end{vmatrix}$ then $A^{-}=\begin{bmatrix} 1-i & 3 \\ i & -i+2 \\ 1 & -3i-2 \end{bmatrix}$

Properties of Transpose of Conjugate of a matrix:

(1) $\left(A^{-}\right)^{-}=A$

(2) $(A+B)^{-}=A^{-}+B^{-}$

(3) $(AB)^{-}=B^{-}.A^{-}$

Hermitian matrix:

A square matrix A is called Hermitian matrix if A= $\bar{A}$ i.e. A= $A = [a_{ij}]_{m\times n}$ is Hermitian if $a_{ij} = \bar{a}_{ji}$ $\forall$ i and j.

Example 10:

Show that the matrix $A=\begin{bmatrix} 1 & 2-i & 3-i \\ 2+i & 3 & -i \\ 3+i & i & 3 \end{bmatrix}$ is Hermitian

Solution:

Here $A=\begin{bmatrix} 1 & 2-i & 3-i \\ 2+i & 3 & -i \\ 3+i & i & 3 \end{bmatrix}$

$\bar{A}=\begin{bmatrix} 1 & 2+i & 3+i \\ 2-i & 3 & i \\ 3-i & -i & 3 \end{bmatrix}$

$\bar{A}' =\begin{bmatrix} 1 & 2-i & 3-i \\ 2+i & 3 & -i \\ 3+i & i & 3 \end{bmatrix}$

$\therefore A = \bar{A}'$

Hence by definition A is Hermitian matrix.

Skew Hermitian Matrix:

A Square matrix A such that $\bar{A}' = -A$ is called a Skew Hermitian Matrix.
i.e. if $A=[a_{ij}]_{m\times n}$ is Skew Hermitian if $a_{ij} = -\bar{a}_{ji}$ $\forall$ i and j.

Here a_{ij} = purely imaginary or re a_{ij} = 0.

Example 11:

Show that the matrix $A=\begin{bmatrix} 2i & 5+i & 6+i \\ -5+i & 0 & -i \\ -6+i & -i & 0 \end{bmatrix}$ is called a Skew Hermitian Matrix.

Solution:

Here $A=\begin{bmatrix} 2i & 5+i & 6+i \\ -5+i & 0 & -i \\ -6+i & -i & 0 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} -2i & 5-i & 6-i \\ -5-i & 0 & i \\ -6-i & i & 0 \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} -2i & -5-i & -6-i \\ 5-i & 0 & i \\ 6-i & i & 0 \end{bmatrix}$$

$$A^{\theta} = -\begin{bmatrix} 2i & 5+i & 6+i \\ -5+i & 0 & -i \\ -6+i & -i & 0 \end{bmatrix}$$

$\therefore$ *Hence* $A^{\theta} = -A$

$\therefore$ The matrix A is Skew Hermitian Matrix.

Note:

Let A be a square matrix expressed as B+iC where B and C are Hermitian and Skew Hermitian Matrices respectively.

$$A = \left[\frac{1}{2}\left(A + A^{\theta}\right)\right] + i\left[\frac{1}{2i}\left(A - A^{\theta}\right)\right] = B + iC$$

$$B = \frac{1}{2}\left(A + A^{\theta}\right) \text{ and } C = \frac{1}{2i}\left(A - A^{\theta}\right)$$

Unitary Matrix:

A square matrix A is said to be unitary matrix if $A^{\theta} A = 1$

Example 12:

Show that the matrix $A = \frac{1}{\sqrt{15}}\begin{bmatrix} -1+3i & -2-i \\ 1-2i & -3-i \end{bmatrix}$ is Unitary matrix.

Solution:

Here $A = \frac{1}{\sqrt{15}}\begin{bmatrix} -1+3i & -2-i \\ 1-2i & -3-i \end{bmatrix}$

$$A^{\theta} = \frac{1}{\sqrt{15}}\begin{bmatrix} -1-3i & 1+2i \\ -2+i & -3+i \end{bmatrix}$$

$$AA^{\theta} = \frac{1}{15}\begin{bmatrix} -1+3i & -2-i \\ 1-2i & -3-i \end{bmatrix}\begin{bmatrix} -1-3i & 1+2i \\ -2+i & -3+i \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore AA^{-} = I$

$\therefore$ Hence A is Unitary Matrix.

Example 13:

Express the matrix, $A = \begin{vmatrix} 2+i & 1 & 3-3i \\ i & 1-i & 2+i \\ 1+i & -3 & 5 \end{vmatrix}$ As the Hermitian Matrix and Skew Hermitian Matrix.

Solution:

Let $\quad A = \begin{vmatrix} 2+i & 1 & 3-3i \\ i & 1-i & 2+i \\ 1+i & -3 & 5 \end{vmatrix}$ ......(I)

$$\bar{A} = \begin{vmatrix} 2-i & 1 & 3+3i \\ -i & 1+i & 2-i \\ 1-i & -3 & 5 \end{vmatrix}$$

$$A^{-} = \begin{vmatrix} 2+i & -1 & 1-i \\ 1 & 1+i & -3 \\ 3+3i & 2-i & 5 \end{vmatrix} ......(II)$$

Adding I and II we get

$$A + A^{-} = \begin{vmatrix} 2+i & 1 & 3-3i \\ i & 1-i & 2+i \\ 1+i & -3 & 5 \end{vmatrix} + \begin{vmatrix} 2+i & -i & 1-i \\ 1 & 1+i & -3 \\ 3+3i & 2-i & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1-i & 4-4i \\ i+1 & 2 & i-1 \\ 4+4i & -i-1 & 10 \end{vmatrix}$$

$$B = \frac{1}{2}\left(A + A^{-}\right) = \frac{1}{2}\begin{vmatrix} 4 & 1-i & 4-4i \\ i+1 & 2 & i-1 \\ 4+4i & -i-1 & 10 \end{vmatrix} ......(III)$$

also $\quad \left(A - A^{-}\right) = \begin{vmatrix} 2+i & 1 & 3-3i \\ i & 1-i & 2+i \\ 1+i & -3 & 5 \end{vmatrix} - \begin{vmatrix} 2+i & -i & 1-i \\ 1 & 1+i & -3 \\ 3+3i & 2-i & 5 \end{vmatrix}$

$$= \begin{vmatrix} 2i & 1+i & 2-2i \\ i-1 & -2 & 5+i \\ -2-2i & -5-i & 0 \end{vmatrix}$$

$$\frac{1}{2}\left(A - \bar{A}\right) = \frac{1}{2}\begin{vmatrix} 2i & 1+i & 2-2i \\ i-1 & -2 & 5+i \\ -2-2i & -5-i & 0 \end{vmatrix} ......(IV)$$

Now, A=B+iC

$$A = \frac{1}{2}\begin{vmatrix} 4 & 1-i & 4-4i \\ i+1 & 2 & i-1 \\ 4+4i & -i-1 & 10 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 2i & 1+i & 2-2i \\ i-1 & -2 & 5+i \\ -2-2i & -5-i & 0 \end{vmatrix}$$

Example 14:

Prove that the matrix, $A = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & i \\ -i & -1 \end{vmatrix}$

Solution:

Let $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$

$$\bar{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

$$\bar{A}A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1-i^2 & i-i \\ -i+i & -i^2+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\therefore A\bar{A} = I$$

Hence A is Unitary.

Check Your Progress:

(1) Show that the following matrices are Skew –Hermitian.

(i) $A = \begin{bmatrix} 2i & 2 & -3 \\ -2 & 4i & -6 \\ 3 & 6 & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 4i & 1+i & 2+2i \\ i-1 & i & 5i \\ 2-2i & -5i & 3i \end{bmatrix}$

(2) Show that the following matrices are Unitary matrices.

(i) $A=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ i-1 & -1 \end{bmatrix}$ (ii) $A=\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ i+1 & 1-i \end{bmatrix}$

(3) If A is Hermitian matrix, then show that iA is Skew- Hermitian matrix.

4.8 LET US SUM UP

In this chapter we have learn

- Cayley Hamilton theorem & it application like Higher power of matrix & Inverse of matrix.
- Minimal .polynomial & derogatory & non-derogatory matrix.
- Complex matrix.
- Hermitian matrix. i.e $A=\bar{A}$
- Skew Hermitian matrix. i.e $\bar{A}=-A$
- Unitary matrix= $A\bar{A}=I$.

4.9 UNIT END EXERCISE

1. Show that the given matrix A satisfies its characteristics equation.

i) $A=\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

ii) $A=\begin{bmatrix} 2 & 4 & 3 \\ 0 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$

iii) $A=\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

2. Using Cayley Hermitian theorem find inverse of the matrix A.

i) $A=\begin{bmatrix} -2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

ii) $A=\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

3. Calculate A^5 by Cayley Hamilton Theorem if $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

4. Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ 1 & -2 & 0 \end{bmatrix}$. Find a similarity transformation that diagonalises matrix A.

5. Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ Find matrix P such that is diagonal matrix.

6. Diagonalise the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

7. For the matrix $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$.

Determine a matrix P such that is diagonal matrix.

8. If show that is Hermitian matrix.

9. Show that the following matrix are skew Hermitian matrix.

i) $A = \begin{bmatrix} 2i & -3 & 4 \\ 3 & 3i & -5 \\ -4 & 5 & 4i \end{bmatrix}$

ii) $= \begin{bmatrix} 0 & 1-i & 2+3i \\ -1-i & 0 & 6i \\ -2+3i & 6i & 0 \end{bmatrix}$

10. Show that the following matrix are unitary matrix

i) $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

ii) $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & w & w^2\\ 1 & w^2 & w\end{bmatrix}$

11. Prove that a real matrix is unitary if it is orthogonal.

12. Check whether the following matrix is derogatory or non-derogatory.

i) $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$

ii) $A=\begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}$

iii) $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{bmatrix}$

iv) $A=\begin{bmatrix}2 & 0 & 1\\ 0 & 3 & 0\\ 1 & 0 & 2\end{bmatrix}$

v) $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$

vi) $A=\begin{bmatrix}5 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 5\end{bmatrix}$

vii) $A=\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$

viii) $A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$

ix) $A=\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}$

13. Show that the following matrix is derogatory also find minimal polynomial.

i) $A=\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

ii) $A=\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

iii) $A=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

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5

VECTOR CALCULAS

UNIT STRUCTURE

1. Objectives
2. Introduction
3. Vector differentiation
4. Vector operator ∇
    1. Gradient
    2. Geometric meaning of gradient
    3. Divergence
    4. Solenoidal function
    5. Curl
    6. Irrational field
5. Properties of gradient, divergence and curl
6. Let Us Sum Up
7. Unit End Exercise

5.0 OBJECTIVES

After going through this unit, you will be able to

- Learn vector differentiation.
- Operators, del, grad and curl.
- Properties of operators

5.1 INTRODUCTION

Vector algebra deals with addition, subtraction and multiplication of vertex. In vector calculus we shall study differentiation of vectors functions, gradient, divergence and curl.

Vector:
Vector is a physical quantity which required magnitude and direction both.

Unit Vector:

Unit Vector is a vector which has magnitude 1. Unit vectors along co-ordinate axis are $\hat{i}$ and $\hat{j}$, $\hat{k}$ respectively.

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$

Scalar Triple Vector:

Scalar triple product of three vectors is defined as $\bar{a}.(\bar{b}\times\bar{c})$ or $[\bar{a}\ \bar{b}\ \bar{c}]$.

Geometrical meaning of $[\bar{a}\ \bar{b}\ \bar{c}]$ is volume of parallelepiped with cotter minus edges $\bar{a}, \bar{b}$ and $\bar{c}$.

We have,

$$[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$$

$$[\bar{a}\ \bar{b}\ \bar{c}] = -[\bar{b}\ \bar{a}\ \bar{c}]$$

Vector Triple Product:

Vector triple product of $\bar{a}\ \bar{b}$ and $\bar{c}$ is cross product of $\bar{a}$ and $(\bar{b}\times\bar{c})$ i.e. $\bar{a}\times(\bar{b}\times\bar{c})$ or cross product of $(\bar{a}\times\bar{b})$ and $\bar{c}$

$$\therefore\ \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\,.\,\bar{c})\,\bar{b} - (\bar{a}\,.\,\bar{b})\,\bar{c}$$

$$(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\,.\,\bar{c})\,\bar{b} - (\bar{b}\,.\,\bar{c})\,\bar{a}$$

Remark : Vector triple product is not associative in general

i.e. $\therefore\ \bar{a}\times(\bar{b}\times\bar{c}) \neq (\bar{a}\times\bar{b})\times\bar{c}$

Coplanar Vectors:

Three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar if $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ for $|\bar{a}| \neq 0, |\bar{b}| \neq 0, |\bar{c}| \neq 0$

5.2 VECTORS DIFFERENTIATION

Let $\bar{v}$ be a vector function of a scalar t. Let $\partial\bar{v}$ be the small increment in a corresponding to the increment ∂t in t.

Then,

$$\partial\bar{v} = \bar{v}(t + \partial t) - \bar{v}(t)$$

$$\frac{\partial\bar{v}}{\partial t} = \frac{\bar{v}(t + \partial t) - \bar{v}(t)}{\partial t}$$

Taking limit $\partial t \longrightarrow 0$ we get,

$$\lim_{\partial t \to 0} \frac{\partial\bar{v}}{\partial t} = \lim_{\partial t \to 0} \frac{\bar{v}(t + \partial t) - \bar{v}(t)}{\partial t}$$

$$\frac{d\bar{v}}{dt} = \lim_{\partial t \to 0} \frac{\partial\bar{v}}{\partial t} = \lim_{\partial t \to 0} \frac{\bar{v}(t + \partial t) - \bar{v}(t)}{\partial t}$$

$$\frac{d\bar{v}}{dt} = \lim_{\partial t \to 0} \frac{\bar{v}(t + \partial t) - \bar{v}(t)}{\partial t}$$

Formulas of vector differentiation:

(i) $\frac{d}{dt} = (k\,\bar{v}) = k\frac{d\bar{v}}{dt}\left[\because \text{ k is a constant}\right]$

(ii) $\frac{d}{dt}(\bar{u} + \bar{v}) = \frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt}$

(iii) $\frac{d}{dt}(\bar{u} \cdot \bar{v}) = \bar{u} \cdot \frac{d\bar{v}}{dt} + \bar{v} \cdot \frac{d\bar{u}}{dt}$

(iv) $\frac{d}{dt}(\bar{u} \times \bar{v}) = \bar{u} \times \frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt} \times \bar{v}$

(v) If $\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Then, $\frac{d\bar{v}}{dt} = \frac{dv_1}{dt}\hat{i} + \frac{dv_2}{dt}\hat{j} + \frac{dv_3}{dt}\hat{k}$

Note:

If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

Example 1:

If $\bar{r} = (t + 1)\,\hat{i} + (t^2 + t - 1)\,\hat{j} + (t^2 - t + 1)\,\hat{k}$ find $\frac{d\bar{r}}{dt}$ and $\frac{d^2\bar{r}}{dt}$

Solution:–

$$\bar{r} = (t+1)\,\hat{i} + (t^2 + t - 1)\,\hat{j} + (t^2 - t + 1)\,\hat{k}$$

$$\frac{d\bar{r}}{dt} = \hat{i} + (2t+1)\,\hat{j} + (2t-1)\,\hat{k}$$

$$\frac{d^2\bar{r}}{dt^2} = 2\hat{j} + 2\,\hat{k}$$

Example 2:

If $\bar{r} = \bar{a}\cos wt + \bar{b}\sin wt$ where w is constant show that

$$\bar{r} \times \frac{d\bar{r}}{dt} = w\,(\bar{a} \times \bar{b}) \text{ and } \frac{d^2\bar{r}}{dt^2} = -w\,\bar{r}$$

Solution: –

$$\bar{r} = \bar{a}\cos wt + \bar{b}\sin wt \text{------------ (i)}$$

$$\frac{d\bar{r}}{dt} = \bar{a}\cos wt + \bar{b}\sin wt \text{------------ (ii)}$$

$$\therefore\ \bar{r} \times \frac{d\bar{r}}{dt} = (\bar{a}\cos wt + \bar{b}\sin wt) \times (-\bar{a}w\sin wt + \bar{b}w\cos wt)$$

$$= (\bar{a} \times \bar{b})\,w\cos^2 wt - (\bar{b} \times \bar{a})\,w\sin^2 wt \qquad \left[\because \bar{a} \times \bar{a} = \bar{0},\ \bar{b} \times \bar{b} = 0\right]$$

$$= (\bar{a} \times \bar{b})\,w\cos^2 wt + (\bar{a} \times \bar{b})\,w\sin^2 wt \qquad \left[\because \bar{b} \times \bar{a} = \bar{0} = -\bar{a} \times \bar{b}\right]$$

$$= (\bar{a} \times \bar{b})\,w\left[\cos^2 wt + \sin^2 wt\right]$$

$$= (\bar{a} \times \bar{b})\,w\,(1)$$

$$= w\,(\bar{a} \times \bar{b})$$

Again differentiating eq^n (ii) w.r.t. „t"

$$\frac{d^2\bar{r}}{dt^2} = -\bar{a}\,w^2\cos wt - \bar{b}\,w^2\sin wt$$

$$= -w^2\,(\bar{a}\cos wt + \bar{b}\sin wt)$$

$$= -w^2 r \text{ from (i)}$$

Example 3. Evaluate the following:

i) $\frac{d}{dt} = [\bar{a}\ \ \bar{b}\ \ \bar{c}]$ ii) $\frac{d}{dt} = \left[\bar{a}\ \ \frac{d\bar{a}}{dt}\ \ \frac{d^2\bar{a}}{dt^2}\right]$

Solution: – i) $\frac{d}{dt} = \left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$

$$= \frac{d}{dt}\left[\bar{a} \,.\, \left(\bar{b} \times \bar{c}\right)\right]$$

$$= \bar{a} \,.\, \frac{d}{dt}\left(\bar{b} \times \bar{c}\right) + \left(\bar{b} \times \bar{c}\right) . \frac{d\bar{a}}{dt}$$

$$= \bar{a} \,.\, \left(\bar{b} \times \frac{d\bar{c}}{dt} + \frac{d\bar{b}}{dt} \times c\right) + \left(\bar{b} \times \bar{c}\right) . \frac{d\bar{a}}{dt}$$

$$= a \,.\, \left(b \times \frac{d\bar{c}}{dt}\right) + a \,.\, \left(\frac{d\bar{b}}{dt} \times \bar{c}\right) + \left(\bar{b} \times \bar{c}\right) . \frac{da}{dt}$$

$$= \left[a \quad b \quad \frac{d\bar{c}}{dt}\right] + \left[\bar{a} \quad \frac{d\bar{b}}{dt} \quad \bar{c}\right] + \left[\bar{b} \quad \bar{c} \quad \frac{d\bar{a}}{dt}\right]$$

Solution: – ii) $\frac{d}{dt} = \left[a \quad \frac{d\bar{a}}{dt} \quad \frac{d^2\bar{a}}{dt^2}\right]$

$$= \left[\bar{a} \quad \frac{d\bar{a}}{dt} \quad \frac{d^3\bar{a}}{dt^3}\right] + \left[\bar{a} \quad \bar{c} \quad \frac{d^2\bar{a}}{dt^2} \quad \frac{d^2\bar{a}}{dt^2}\right] + \left[\frac{d\bar{a}}{dt} \quad \frac{d^2\bar{a}}{dt^2} \quad \frac{d\bar{a}}{dt}\right]$$

(From Result i)

$$= \left[\bar{a} \quad \frac{d\bar{a}}{dt} \quad \frac{d^3\bar{a}}{dt^3}\right] + 0 + 0$$

$$= \left[\bar{a} \,. \quad \frac{d\bar{a}}{dt} \quad \frac{d^3\bar{a}}{dt^3}\right]$$

Example 4. Evaluate the following: $\frac{d}{dt} = \left[\left(\bar{a} \times \bar{b}\right) \times \bar{c}\right]$

Solution: $\frac{d}{dt} = \left[\left(\bar{a} \times \bar{b}\right) \times \bar{c}\right]$

$$= \left(a \times \bar{b}\right) \times \frac{d\bar{c}}{dt} + \frac{d}{dt}\left(\bar{a} \times \bar{b}\right) \times c$$

$$= \left(a \times \bar{b}\right) \times \frac{d\bar{c}}{dt} + \left(\bar{a} \times \frac{d\bar{b}}{dt} + \frac{d\bar{a}}{dt} \times \bar{b}\right) \times c$$

$$= \left(a \times \bar{b}\right) \times \frac{d\bar{c}}{dt} + \left(a \times \frac{db}{dt}\right) \times c + \left(\frac{da}{dt} \times b\right) \times c$$

Example 5. Show that $\hat{r} \times \frac{d\hat{r}}{dt} = \frac{\hat{r} \times \frac{d\hat{r}}{dt}}{r^2}$, where $\hat{r} = \frac{\bar{r}}{r}$

Solution : We have $\hat{r} = \frac{\bar{r}}{r}$

$$\therefore \frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\frac{\bar{r}}{r}\right)$$

$$= \frac{r\frac{d\bar{r}}{dt} - \bar{r}\frac{dr}{dt}}{r^2}$$

$$= \frac{1}{r}\frac{d\bar{r}}{dt} - \frac{\bar{r}}{r^2}\frac{dr}{dt}$$

L.H.S. $\hat{r} = \frac{\bar{r}}{r}$

$$= \frac{\bar{r}}{r} \times \left(\frac{1}{r}\frac{d\bar{r}}{dt} - \frac{\bar{r}}{r^2}\frac{dr}{dt}\right)$$

$$= \frac{\bar{r}}{r} \times \frac{1}{r}\frac{d\bar{r}}{dt} - \frac{\bar{r}\times\bar{r}}{r^2}\frac{dr}{dt}$$

$$= \frac{\bar{r}}{r^2} \times \frac{d\bar{r}}{dt} - \bar{0} \qquad [\because \bar{r}\times\bar{r} = 0]$$

$$= \frac{\bar{r} \times \frac{d\bar{r}}{dt}}{r^2}$$

= R.H.S

Example 6. If $\bar{r} = t^3 i + \left(2t_3 - \frac{1}{5t^2}\right) j$. Then show that $\bar{r} \times \frac{d\bar{r}}{dt} = \hat{k}$

Solution:

$$\bar{r} = t^3 i + \left(2t^3 - \frac{1}{5t^2}\right) j$$

$$\frac{d\bar{r}}{dt} = 3t^2 i + \left(6t^2 + \frac{2}{5t^3}\right) j$$

L.H.S.

$$\bar{r} \times \frac{d\bar{r}}{dt} = \begin{vmatrix} i & j & k \\ t^3 & 2t^3 - \frac{1}{5t^2} & 0 \\ 3t^2 & 6t^2 + \frac{2}{5t^3} & 0 \end{vmatrix}$$

$$= i\,(0) - j(0) + k\left[t^3\left(6t^2+\frac{2}{5t^3}\right) - 3t^2\left(2t^3-\frac{1}{5t^2}\right)\right]$$

$$= k\left[\left(6t^5+\frac{2}{5}-6t^5+\frac{3}{5}\right)\right]$$

$$= \hat{k}$$

$$= \text{R. H. S.}$$

Example 7. If $\bar{r} = \bar{a}\, e^{mt} + \bar{b}\, e^{-mt}$. Show that $\frac{d^2\bar{r}}{dt^2} = n^2\bar{r}$

Solution:

$\bar{r} = \bar{a}\, e^{mt} + \bar{b}\, e^{-mt}$ ....(i)

$$\frac{d\bar{r}}{dt} = m\,\bar{a}\, e^{mt} - m\,\bar{b}\, e^{-mt}$$

$$\frac{d^2\bar{r}}{dt^2} = m^2\,\bar{a}\, e^{mt} + m^2\,\bar{b}\, e^{-mt}$$

$$= m^2\left(\bar{a}\, e^{mt} + \bar{b}\, e^{-mt}\right)$$

$$= m^2\,\bar{r} \qquad \text{(from (i))}$$

$$\frac{d^2\bar{r}}{dt^2} = m^2\,\bar{r}$$

Check your progress:

(1) If $\frac{d\bar{u}}{dt} = \bar{w}\times\bar{u}$ and $\frac{d\bar{v}}{dt} = \bar{w}\times\bar{v}$

Show that $\frac{d}{dt}(\bar{u}\times v) = \bar{w}\times(\bar{u}\times v)$

(2) If $\bar{r} = t^2 i + (3t^3 - t^2)\, j + (7t+1)\,\hat{k}$ Find $\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}$

(3) If: $\bar{r} = t\,\hat{i} - t\,\hat{j} + (st-1)\,\hat{k}$, Find $\frac{d\bar{r}}{dt}, \frac{d^2\bar{r}}{dt^2}, \left|\frac{d\bar{r}}{dt}\right|, \left|\frac{d^2\bar{r}}{dt^2}\right|$

(4) If $\bar{r} = e^t\, i + (2\cos 3t)\, j + (7\sin 3t)\,\hat{j}$ Find $\frac{d^2\bar{r}}{dt^2}$ at $t = \frac{®}{2}$

(5) Show that: $\bar{a}\cdot\frac{d\bar{a}}{dt} = a\frac{da}{dt}$ where $a = a_1 i + a_2 j + a_3 k$ and a is magnitude of $\bar{a}$.

5.3 VECTOR OPERATOR

The vector differential operator ∇ is defined as $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$.

5.3.1 Gradient:

The gradient of a scalar function is denoted by grad ¤ or ∇¤ and is defined as ∇¤ $= \hat{i}\frac{\partial ¤}{\partial x} + \hat{j}\frac{\partial ¤}{\partial y} + \hat{k}\frac{\partial ¤}{\partial z}$. Note that grad ¤ is a vector quantity.

5.3.2 Geometric meaning of gradient:

The grad ¤ is a vector right angled to the surface, whose equation is ¤ $(x, y, z) = c$, where c is constant.

Hence for $\bar{r} = x\,i + y\,j + z\,k$ any point on surface $\therefore$ ∇¤ $. d\bar{r} = 0$

i.e. ∇¤ at is right angles to $d\bar{r}$ and $d\bar{r}$ lies on the tangent plane to the surface at $P(\bar{r})$.

$\therefore$ ∇¤ $\perp d\bar{r}$

Geometrically ∇¤ represents a vector normal to the surface $\phi(x, y, z)$ = constant.

Example 8: Find grad ϕ, where $\phi = x^2 y^3 e^z$

Solution: grad $\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 y^3 e^z)$

$$= \hat{i}\frac{\partial}{\partial x}(x^2 y^3 e^z) + \hat{j}\frac{\partial}{\partial y}(x^2 y^3 e^z) + \hat{k}\frac{\partial}{\partial z}(x^2 y^3 e^z)$$

$$= \hat{i}(2xy^3e^z) + \hat{j}(3x^2 y^2 e^z) + \hat{k}(x^2 y^3 e^z)$$

$$= x\,y^2e^z\,(2y\,\hat{i} + 3x\hat{j} + xy\hat{k})$$

Example 9: If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ find grad r

Solution:

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = \sqrt{x^2 + y^2 + z^2}$

$$\text{Grad } r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\sqrt{x^2 + y^2 + z^2}$$

$$= \hat{i}\frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} + \hat{j}\frac{\partial}{\partial y}\sqrt{x^2 + y^2 + z^2} + \hat{k}\frac{\partial}{\partial z}\sqrt{x^2 + y^2 + z^2}$$

$$= \hat{i}\frac{1}{2\sqrt{x^2 + y^2 + z^2}}(2x) + \hat{j}\frac{1}{2\sqrt{x^2 + y^2 + z^2}}.(2y) + \hat{k}\frac{1}{2\sqrt{x^2 + y^2 + z^2}}(2z)$$

$$= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$$\therefore \text{ grad } r = \frac{\bar{r}}{r}$$

Example 10: If $r = x\hat{i} + y\hat{j} + z\hat{k}$ find grad $\frac{1}{r}$

Solution:

$r = x\hat{i} + y\hat{j} + z\hat{k}$

$r = \sqrt{x^2 + y^2 + z^2}$

$\therefore r^2 = x^2 + y^2 + z^2$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{2r}{2x} = \frac{x}{r}, \quad \frac{2r}{2y} = \frac{y}{r}, \quad \frac{2z}{2r} = \frac{z}{r}$$

$$\text{grad } \frac{1}{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(\frac{-1}{r_2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r_2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(\frac{-1}{r_2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left[i\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial y}\right]$$

$$= \frac{-1}{r^2}\left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right)$$

$$= \frac{-1}{r^2}\cdot\frac{-1}{r}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{-1}{r^3}\, r$$

$$= \frac{-r}{r^3}$$

Example 11: If $\phi = 2x^3y - y^2z$ find grad ϕ at (1, -1, 2)

Solution:

$$\text{grad } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(2x^3y - y^2 z\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(2x^3y - y^2z\right) + \hat{j}\frac{\partial}{\partial y}\left(2x^3y - y^2 z\right) + \hat{k}\frac{\partial}{\partial z}\left(2x^3y - y^2 z\right)$$

$$= \hat{i}\left(6x^2y\right) + \hat{j}\left(2x^3 - 2yz\right) + \hat{k}\left(-y^2\right)$$

$$= \hat{i}\, 6x^2y + \hat{j}\left(2x^3 - 2yz\right) - \hat{k}\, y^2$$

At (1, -1, and 2)

$$\text{grad } \phi = 6\,(1)^2(-1)\, i + j\left(2(1)^3 - 2(-1)(2)\right) - \hat{k}\,(-1)^2$$

$$= 6\, i + j\,(2+4) - \hat{k}$$

$$= -6\, i + 6j - \hat{k}$$

Example 12: Evaluate grad e^{r^2}, where $r^2 = x^2 + y^2 + z^2$

Solution : Grad $\left(e^{r^2}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) e^{r^2}$

$$= \hat{i}\frac{\partial}{\partial x}\left(e^{r^2}\right) + \hat{j}\frac{\partial}{\partial y}\left(e^{r^2}\right) + \hat{k}\frac{\partial}{\partial z}\left(e^{r^2}\right)$$

$$= \hat{i}\, e^{r^2}.\partial r\frac{\partial r}{\partial x} + \hat{j}\, e^{r^2}.\partial r\frac{\partial r}{\partial y} + \hat{k}\, e^{r^2}.\partial r\frac{\partial r}{\partial z}$$

$$= \hat{i}\, e^{r^2}.\partial r.\frac{x}{r} + \hat{j}\, e^{r^2}.\partial r\frac{y}{r} + \hat{k}\, e^{r^2}.\partial r\frac{z}{r}$$

$$= r\, e^{r^2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= r\, e^{r^2}\,\bar{r}$$

Example 13: Find grad r^n

Solution: grad $r^n = \nabla r^n$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) r^n$$

$$= \hat{i} \frac{\partial}{\partial x} r^n + \hat{j} \frac{\partial}{\partial y} r^n + \hat{k} \frac{\partial}{\partial z} r^n$$

$$= \hat{i}\, n\, r^{n-1} \frac{\partial r}{\partial x} + \hat{j}\, n\, r^{n-1} \frac{\partial r}{\partial y} + \hat{k}\, n\, r^{n-1} \frac{\partial r}{\partial z}$$

$$= \hat{i}\, n\, r^{n-1} \frac{x}{r} + \hat{j}\, n\, r^{n-1} \frac{y}{r} + \hat{k}\, n\, r^{n-1} \frac{z}{r}$$

$$= \hat{i}\, n\, r^{n-2} x + \hat{j}\, n\, r^{n-2} y + \hat{k}\, n\, r^{n-2} z$$

$$= n\, r^{n-2} \left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$= n\, r^{n-2} \bar{r}$$

Example 14: Find grad $\log\left(x^2 + y^2 + z^2\right)$

Solution:

$$\text{grad } \log\left(x^2 + y^2 + z^2\right) = \text{grad } \log r^2 = \text{grad } (2 \log r) = 2 \text{ grad } (\log r)$$

$$= 2 \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (\log r)$$

$$= 2 \left(\hat{i} \frac{\partial}{\partial x} (\log r) + \hat{j} \frac{\partial}{\partial y} (\log r) + \hat{k} \frac{\partial}{\partial z} (\log r) \right)$$

$$= 2 \left(\hat{i} \frac{1}{r} \frac{\partial r}{\partial x} + \hat{j} \frac{1}{r} \frac{\partial r}{\partial y} + \hat{k} \frac{1}{r} \frac{\partial r}{\partial z} \right)$$

$$= 2 \left(\hat{i} \frac{1}{r} \frac{x}{r} + \hat{j} \frac{1}{r} \frac{y}{r} + \hat{k} \frac{1}{r} \frac{z}{r} \right)$$

$$= \frac{2}{r^2} \left(x\hat{i} + y\hat{j} \frac{1}{r} \frac{y}{r} + z\hat{k} \right)$$

$$= \frac{2\bar{r}}{r^2}$$

Example 15: Show that $\text{grad} \left(\frac{\bar{a} \cdot \bar{r}}{r^n} \right) = \frac{\bar{a}}{r^n} - \frac{n(\bar{a} \cdot \bar{r})}{r^{n+2}}\, \bar{r}$ where $\bar{r} = x\,i + yj + zk$

Solution: let

$\bar{a} = a_1 i + a_2 j + a_3 k$

$\therefore \bar{a} . \bar{r} = a_1 x + a_2 y + a_3$

$\therefore \text{grad} \left(\frac{\bar{a}. \bar{r}}{r^n} \right)$

$= \nabla \left(\frac{\bar{a}. \bar{r}}{r^n} \right)$

$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \left(\frac{a_1 x + a_2 y + a_3 z}{r^n} \right)$

now $\therefore i \frac{\partial}{\partial x} \left(\frac{a_1 x + a_2 y + a_3 z}{r^n} \right)$

$= i \left(\frac{r^n a_1 - (a_1 x + a_2 y + a_3 z) n r^{n-1} \frac{\partial r}{\partial x}}{r^{2n}} \right)$

$= i \left(\frac{r^n a_1 - (a_1 x + a_2 y + a_3 z) n r^{n-1} \frac{x}{r}}{r^{2n}} \right)$

$= i \left(\frac{r^n a_1 - (a_1 x + a_2 y + a_3 z) n x^{n-1} r^{n-2}}{r^{2n}} \right)$

similarly

$= \hat{j} \frac{\partial}{\partial y} \left(\frac{(a_1 x + a_2 y + a_3 z)}{r^n} \right)$

$= \hat{j} \left(\frac{r^n a_2 - (a_1 x + a_2 y + a_3 z) n y r^{n-2}}{r^{2n}} \right)$

and

$= \hat{k} \frac{\partial}{\partial z} \left(\frac{(a_1 x + a_2 y + a_3 z)}{r^n} \right)$

$= \hat{k} \left(\frac{r^n a_3 - (a_1 x + a_2 y + a_3 z) n z r^{n-2}}{r^{2n}} \right)$

$\therefore \text{grad} \left(\frac{\bar{a} . \bar{r}}{r^n} \right)$

$= \frac{r_n (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) - (a_1 x + a_2 y + a_3 z) n z r^{n-2} (x\hat{i} + y\hat{j} + z\hat{k})}{r^{2n}}$

$$= \frac{\bar{a}r^n - n\,r^{n-2}\,\bar{r}\,(\bar{a}.\bar{r})}{r^{2n}}$$

$$= \frac{\bar{a}r^n}{r^{2n}} - \frac{n(\bar{a}.\bar{r})r^{n-2}\,\bar{r}}{r^{2n}}$$

$$= \frac{\bar{a}r^n}{r^{2n}} - \frac{n(\bar{a}.\bar{r})}{r^{n+2}}\,\bar{r}$$

$$= \frac{\bar{a}}{r^n} - \frac{n(\bar{a}.\bar{r})}{r^{n+2}}\,\bar{r}$$

Check your progress:

(1) If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\bar{r}|$

Show that:

a) $\text{grad}\,(\log r) = \frac{\bar{r}}{r^2}$

b) $\text{grad}\,r^3 = 3r\bar{r}$

c) $\text{grad}\,f(r) = f^1(r)\,\frac{\bar{r}}{r}$

(2) If $\phi = 4x^2yz + 3xyz^2 - 5xyz$

Find grad ϕ at (3, 2, -1)

(3) Show that $\text{grad}\,r^3 = -3\,r^{-5}\,\bar{r}$

(4) If $F(x, y, z) = x^2 + y^2 + z^2$ Find ∇F at (1, 1, 1)

(5) Show that $\nabla f(r) \times \bar{r} = 0$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(6) Find unit vector normal to the surface $x^2 + y^2 + z^2 = 3a^2$ at (a, a, a)

[Hint :- Unit vector normal to surface ϕ i.e. $\frac{\nabla\phi}{|\nabla\phi|}$]

5.3.1 Divergence:

If v (x, y, z) = $v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ can be defined and differentiated at each point (x, y, z) in a region of space then divergence of v is defined as div v = ∇ . v

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) . (v_1\, i + v_2\, j + v_3\, k)$$

$$= \frac{\partial}{\partial x}(v_1) + \frac{\partial}{\partial y}(v_2) + \frac{\partial}{\partial z}(v_3)$$

Example 16 If $\bar{F} = (x^2 - y^2)\,\hat{i} + 2xy\hat{j} + (y^2 - 2xy)\,\hat{k}$, find $\bar{F}$

Solution: div $\bar{F} = \nabla . \bar{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) . \left\{ (x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - 2xy)\hat{k} \right\}$$

$$= \frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(y^2 - 2xy)$$

$$= 2x + 2x + 0$$

$$= 4x$$

Example 17 Show that div $\bar{r} = 3$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution: div $\bar{r}$

$$= \nabla . \bar{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) . (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3$$

Example 18 For $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that div $(r^n\, \bar{r}) = (n+3)\, r^n$ where $r = |\bar{r}|$

Solution: L.H.S. div $(r^n\, \bar{r}) = \nabla . (r^n\, r)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) . r^n (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(r^n\, x) + \frac{\partial}{\partial y}(r^n\, y) + \frac{\partial}{\partial z}(r^n\, z)$$

$$= r^n (1) + x\, nr^{n-1}\frac{\partial r}{\partial x} + r^n (1) + y\, nr^{n-1}\frac{\partial r}{\partial y} + r^n (1) + z\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right)$$

$$= 3r^n + nr^{n-1}\left(x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right)$$

$$= 3r^n + nr^{n-1}\frac{\left(x^2 + y^2 + z^2\right)}{r}$$

$$= 3r^n + nr^{n-1}\frac{r^2}{r}$$

$$= 3r^n + nr^n$$

$$= (3 + n)r^n$$

= R.H.S.

Example 19 Evaluate div $(\hat{r})$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution: We have $\hat{r}^n = \frac{\bar{r}}{r}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$\therefore$ div $(\hat{r})$

$$= \nabla.\hat{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2} + \frac{r(1) - y\frac{\partial r}{\partial y}}{r^2} + \frac{r(1) - z\frac{\partial r}{\partial z}}{r^2}$$

$$= \frac{r - x\left(\frac{x}{r}\right)}{r^2} + \frac{r - y\frac{y}{r}}{r^2} + \frac{r - z\frac{z}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} + \frac{r^2 - z^2}{r^3}$$

$$= \frac{r^2 - x^2 + r^2 - y^2 + r^2 - z^2}{r^3}$$

$$= \frac{3r^2 - \left(x^2 + y^2 + z^2\right)}{r^3}$$

$$= \frac{3r^2 - r^2}{r^3}$$

$$= \frac{2}{r}$$

Example 20 If $F = x^2\ y^3\ z^4$ Find div (grad F)

Solution: grad F

$$= \nabla F$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2\ y^3\ z^4\right)$$

$$= 2xy^3z^4\ \hat{i} + 3y^2z^4\ \hat{j} + 4x^2y^3z^3\ \hat{k}$$

$\therefore$ div (grad F)

$$= \nabla.\left(2xy^3z^4\ \hat{i} + 3y^2x^2z^4\ \hat{j} + 4x^2y^3z^3\ \hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(2xy^3z^4\right) + \frac{\partial}{\partial y}\left(3y^2\ x^2\ z^4\right) + \frac{\partial}{\partial z}\left(4\,x^2y^3\ z^3\right)$$

$$= 2xy^3z^4 + 6x^2y\,z^4 + 12\,x^2y^3\ z^2$$

Example 21 Find the value of div $(\bar{a} \times \bar{r})\,r^n$ where $\bar{a}$ is a constant vector and $\bar{r} = x\hat{i} + y\,\hat{j} + z\,\hat{k}$

Solution: div $(\bar{a} \times \bar{r})\,r^n$

$$= \hat{i}\frac{\partial}{\partial x}\,.\left\{(\bar{a} \times \bar{r})\,r^n\right\}$$

$$= \sum \hat{i}\,.\left[\left\{\frac{\partial}{\partial x}\,.\,\{(\bar{a} \times \bar{r})\}\right\}r^n + (\bar{a} \times \bar{r})\frac{\partial}{\partial x}\left(r^n\right)\right]$$

$$= \sum \hat{i}\,.\left[\left(\bar{a} \times \frac{\partial \bar{r}}{\partial x}\right)r^n + (\bar{a} \times \bar{r})\,n\,r^{n-1}\frac{\partial r}{\partial x}\right]$$

$$\left[\because \frac{\partial \bar{a}}{\partial x} = 0\right]$$

$$= \sum \hat{i}\,.\left[\left(\bar{a} \times \hat{i}\right)r^n + (\bar{a} \times \bar{r})\,n\,r^{n-1}\frac{x}{r}\right]$$

$$= \sum \hat{i}\,.\left[\left(\bar{a} \times \hat{i}\right)r^n + nx\,r^{n-2}\,(\bar{a} \times \bar{r})\right]$$

$$= \sum nr^{n-2}\left(x\,\hat{i}\right)\left(\bar{a} \times \bar{r}\right) \qquad \left[\because \hat{i}.\left(\bar{a} \times \hat{i}\right) = 0\right]$$

$$= nr^{n-2}\left(\bar{a} \times \bar{r}\right)\sum x\,\hat{i}$$

$$= nr^{n-2}(\bar{a} \times \bar{r})\,\bar{r} \qquad \left[\because \sum x\hat{i} = \bar{r}\right]$$

$$= nr^{n-2}\left[(\bar{a} \times \bar{r}) \,.\, \bar{r}\right]$$

$$= nr^{n-2}(0)$$

$$= 0$$

5.3.4 Solenoidal Function: A vector function $\bar{F}$ is called Solenoidal if div $\bar{F} = 0$ at all points of the function.

5.3.5 Curl: The curl of a vector point function $\bar{F}$ is defined as curl $\bar{F} = \nabla \times \bar{F}$ if $F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

$$\because \text{curl } \bar{F} = \nabla \times \bar{F}$$

$$= (\nabla \times \bar{F})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

The curl of the linear velocity of any particle of rigid body is equal to twice the angular velocity of body.

i.e. if $\bar{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$ be the angular velocity of any particle of the body with position vector defined as $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then linear velocity $\bar{v} = \bar{w} \times \bar{r}$.

Hence curl $\bar{v} = \nabla \times \bar{v}$

$$= \nabla \times (\bar{w} \times \bar{r})$$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[\hat{i}\,(w_2 z - w_3 y) - \hat{j}\,(w_1 z - w_3 x) + \hat{k}\,(w_1 y - w_2 x)\right]$$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$$

$$= \hat{i}\,(w_1 + w_1) - \hat{j}\,(-w_2 - w_2) + \hat{k}\,(w_3 + w_3)$$

$$= 2w_1 \hat{i} + 2w_2 \hat{j} + 2w_3 \hat{k}$$

$$= 2\bar{w}$$

$$\therefore \text{ curl } \bar{v} = 2\bar{w}$$

5.3.6 Irrotational field:

A vector point function $\bar{F}$ is called irrotational if $\bar{F} = \bar{0}$ at all points of the function.

Example 22 Find curl (curl $\bar{F}$) If $\bar{F} = x^2 y\, \hat{i} - 2 x z\, \hat{j} + 2 y z\, \hat{k}$ at (1, 0, 2)

Solution: Curl $\bar{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xy & 2yz \end{vmatrix}$$

$$= (2z + 2x)\hat{i} + \left(-2z - x^2\right)\hat{k}$$

$$\therefore \text{ curl curl } (\bar{F}) = \nabla \times \left[(2z + 2x)i + 0j + \left(-2z - x^2\right)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z + 2x & 0 & -2z - x^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(-2z - x^2\right) - \frac{\partial}{\partial z}(0)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(-2z - x^2\right) - \frac{\partial}{\partial z}(2z + 2x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(2z + 2x)\right]$$

$$= \hat{i}(0) - \hat{j}\,(-2x - 2) + \hat{k}(0)$$

$$= (2x + 2)\,\hat{j}$$

At (1, 0, 2)

$$\left(\text{curl } \overline{F}\right) = \left[2(1) + 2\right]\hat{j}$$

$$= 4\hat{j}$$

Example 23 Find curl $\overline{V}$ if $\overline{V} = \left(x^2 + yz\right)\hat{i} + \left(y^2 + 2x\right)\hat{j}\ \left(z^2 + xy\right)\hat{k}$

Solution: curl $\overline{V}$

$= \nabla \times \overline{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + yz & y^2 + zx & z^2 + xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(z^2 + xy\right) - \frac{\partial}{\partial z}\left(y^2 + 2x\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(z^2 + xy\right) - \frac{\partial}{\partial z}\left(y^2 + yz\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(y^2 + 2x\right) - \frac{\partial}{\partial y}\left(x^2 + yz\right)\right]$$

$$= \hat{i}\left(x - x\right) - \hat{j}\left(y - y\right) + \hat{k}\left(z - z\right)$$

$$= \overline{0}$$

Example 24 Evaluate curl $\overline{r}$ where if $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution: Curl $\overline{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$= \overline{0}$$

Example 25 Evaluate curl $\left(\frac{\hat{r}}{r}\right)$ where if $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution:

$$\hat{r} = \left(\frac{\overline{r}}{r^2}\right)$$

$$\therefore \frac{\hat{r}}{r} = \frac{x}{r^2}\hat{i} + \frac{y}{r^2}\hat{j} + \frac{z}{r^2}\hat{k}$$

$$\therefore \text{curl}\left(\frac{\hat{r}}{r}\right) = \nabla \times \left(\frac{x}{r^2}\hat{i} + \frac{y}{r^2}\hat{j} + \frac{z}{r^2}\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^2}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{z}{r2}\right) - \frac{\partial}{\partial z}\left(\frac{x}{r2}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{r2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r^2}\right)\right]$$

$$= \hat{i}\left[\frac{-2z}{r^3}\frac{2r}{2y} + \frac{2y}{r^3}\frac{2r}{2z}\right] + \text{-------------} + \text{---------------}$$

$$= \hat{i}\left[\frac{-2z}{r^3}\frac{y}{r} + \frac{2y}{r^3}\frac{z}{r}\right] + \text{-------------} + \text{---------------}$$

$$= \hat{i}\left[\left(\frac{2yz - 2yz}{r^3}\right) + \hat{j}\left(\frac{2zx - 2zx}{r^3}\right) + \hat{k}\left(\frac{2xy - 2xy}{r^3}\right)\right]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \overline{0}$$

Example 25 If $F = x^2y\,\hat{i} + xz\hat{j} + 2yz\hat{k}$ find div (curl F)

Solution: curl F

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}\left(x^2 z\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x\,z) - \frac{\partial}{\partial y}\left(x^2 z\right)\right]$$

$$= \hat{i}\,(2z - x) - \hat{j}(0 - 0) + \hat{k}\left(z - x^2\right)$$

$$= (2z - x)\,\hat{i} + \left(z - x^2\right)\hat{k}$$

div (curl $\overline{F}$)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) . \left[(2z - x)\,\hat{i} + \left(z - x^2\right)\hat{k} \right]$$

$$= \frac{\partial}{\partial x}(2z - x) + \frac{\partial}{\partial z}\left(z - x^2\right)$$

$$= -1 + 1$$

$$= 0$$

Example 27 If $\overline{F} = \text{grad}\,(xy + yz + zx)$, find $(\text{curl}\,\overline{F})$.

Solution: $\overline{F} = \text{grad}\,(xy + yz + zx)$

$$= \nabla\,(xy + yz + zx)$$

$$== \left[i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right](xy + yz + zx)$$

$$= \hat{i}\,\frac{\partial}{\partial x}(xy + yz + zx) + \hat{j}\,\frac{\partial}{\partial y}(xy + yz + zx) + \hat{k}\,\frac{\partial}{\partial z}(xy + yz + zx)$$

$$= \hat{i}\,(y + z) + \hat{j}(x + z) + \hat{k}\,(y + x)$$

$\therefore$ $(\text{curl}\,\overline{F})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z & x + z & x + y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x + y) - \frac{\partial}{\partial z}(x + z) \right] - \hat{j}\left[\frac{\partial}{\partial x}(x + y) - \frac{\partial}{\partial z}(y + z) \right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x + z) - \frac{\partial}{\partial y}(y + z) \right]$$

$$= \hat{i}(1 - 1) - \hat{j}\,(1 - 1) + \hat{k}\,(1 - 1)$$

$$= 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$$

$$= \overline{0}$$

5.4 PROPERTIES OF GRADIENT, DIVERGENCE AND CURL

(i) $\nabla\,(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla .\left(\overline{A} \pm \overline{B}\right) = \nabla.\overline{A} \pm \nabla.\overline{B}$

(iii) $\nabla \times \left(\overline{A} \pm \overline{B}\right) = \nabla \times \overline{A} \pm \nabla.\overline{B}$

Proof:

(i) $\nabla(f \pm g) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(f \pm g)$

$= \hat{i}\frac{\partial}{\partial x}(f \pm g) \pm \hat{j}\frac{\partial}{\partial y}(f \pm g) + \hat{k}\frac{\partial}{\partial z}(f \pm g)$

$= \left(\hat{i}\frac{\partial}{\partial x}f + \hat{j}\frac{\partial}{\partial y}f + \hat{k}\frac{\partial}{\partial z}\right) \pm \left(\hat{i}\frac{\partial}{\partial x}g + \hat{j}\frac{\partial}{\partial y}g + \frac{\partial}{\partial z}g\right)$

$= \nabla f \pm \nabla g$

(ii) Let $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$B = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$\therefore\ \nabla \cdot (\bar{A} \pm \bar{B})$

$= \nabla \cdot \left[(\bar{A}_1 \pm \bar{B}_1)\hat{i} + (\bar{A}_2 \pm \bar{B}_2)\hat{j} + (\bar{A}_3 \pm \bar{B}_3)\hat{k}\right]$

$= \frac{\partial}{\partial x}(\bar{A}_1 \pm \bar{B}_1) + \frac{\partial}{\partial y}(\bar{A}_2 \pm \bar{B}_2) + \frac{\partial}{\partial z}(\bar{A}_3 \pm \bar{B}_3)$

$= \frac{\partial}{\partial x}(A_1) + \frac{\partial}{\partial y}(A_2) + \frac{\partial}{\partial z}(A_3) \pm \left[\frac{\partial}{\partial x}(B_1) + \frac{\partial}{\partial y}(B_2) + \frac{\partial}{\partial z}(B_3)\right]$

$= \nabla \cdot \bar{A} \pm \nabla \cdot \bar{B}$

(ii) Let

$\nabla \times (\bar{A} \pm \bar{B})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \bar{A}_1 \pm \bar{B}_1 & \bar{A}_2 \pm \bar{B}_2 & \bar{A}_3 \pm \bar{B}_3 \end{vmatrix}$$

$= \sum \hat{i}\left[\frac{\partial}{\partial y}(\bar{A}_3 \pm \bar{B}_3) - \frac{\partial}{\partial z}(\bar{A}_2 \pm \bar{B}_2)\right]$

$= \sum \hat{i} \times \frac{\partial}{\partial x}(\bar{A} \pm \bar{B})$

$= \sum \hat{i} \times \left(\frac{\partial \bar{A}}{\partial x} \pm \frac{\partial \bar{B}}{\partial x}\right)$

$= \sum \hat{i} \times \frac{\partial \bar{A}}{\partial x} \pm \sum \hat{i} \times \frac{\partial \bar{B}}{\partial x}$

$= \nabla \times \bar{A} \pm \nabla \times B$

Thank You