Applied Mathematics

Part-1

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MATRICES

UNIT STRUCTURE

1.0 OBJECTIVES

In this chapter a student has to learn the

- Concept of adjoint of a matrix.
- Inverse of a matrix.
- Rank of a matrix and methods finding these.

1. INTRODUCTION

At higher secondary level, we have studied the definition of a matrix, operations on the matrices, types of matrices inverse of a matrix etc.

In this chapter, we are studying adjoint method of finding the inverse of a square matrix and also the rank of a matrix.

2. DEFINITIONS

1) **Definitions:-** A system of $m \times n$ numbers arranged in the form of an ordered set of m horizontal lines called rows & n vertical lines called columns is called an $m \times n$ matrix.
The matrix of order $m \times n$ is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2j} & a_{2n} \\ \ldots\ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & a_{i3} & a_{ij} & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & a_{mj} & a_{mn} \end{bmatrix}_{n\times n}$$

Note:

i) Matrices are generally denoted by capital letters.

ii) The elements are generally denoted by corresponding small letters.

Types of Matrices:

1) Rectangular matrix :-

Any mxn Matrix where $m \neq n$ is called rectangular matrix.

For e.g

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix}_{2\times 3}$$

2) Column Matrix :

It is a matrix in which there is only one column.

$$x = \begin{bmatrix} 1 \\ 2 \\ \\ 4 \end{bmatrix}_{3\times 1}$$

3) Row Matrix :

It is a matrix in which there is only one row.

$$x = \begin{bmatrix} 5 & 7 & 9 \end{bmatrix}_{1\times 3}$$

4) Square Matrix :

It is a matrix in which number of rows equals the number of columns.

i.e its order is n x n.

e.g.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}_{2\times 2}$$

5) Diagonal Matrix:

It is a square matrix in which all non-diagonal elements are

zero. e.g.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

6) Scalar Matrix:

It is a square diagonal matrix in which all diagonal elements are

equal. e.g.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}$$

7) Unit Matrix:

It is a scalar matrix with diagonal elements as unity.

e.g.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

8) Upper Triangular Matrix:

It is a square matrix in which all the elements below the principle diagonal are zero.

e.g.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3}$$

9) Lower Triangular Matrix:

It is a square matrix in which all the elements above the principle diagonal are zero.

e.g.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 3 & 2 \end{bmatrix}_{3\times3}$$

10) Transpose of Matrix:

It is a matrix obtained by interchanging rows into columns or columns into rows.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 7 & 9 \end{bmatrix}_{2\times3}$$

$$A^T = Transpose\ of\ A = \begin{bmatrix} 1 & 3 \\ 3 & 7 \\ 5 & 9 \end{bmatrix}_{3\times2}$$

11) Symmetric Matrix:

If for a square matrix A, $A = A^T$ then A is symmetric

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 1 \\ 5 & 1 & 9 \end{bmatrix}$$

12) Skew Symmeric Matrix :

If for a square matrix A, $A = A^T$ then it is skew -symmetric matrix.

$$A = \begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & 3 \\ -7 & -3 & 0 \end{bmatrix}$$

Note : For a skew Symmetric matrix, diagonal elements are zero.

Determinant of a Matrix:

Let A be a square matrix then

$| \ A \ | = determinant \ of \ \ A \ \ i.e \ det \ A = |A|$

If (i) then $|A| \ \neq 0$ matrix A is called as non-singular and

If (i) then $|A| = 0,$ matrix A is singular.

Note : for non-singular matrix A-1 exists.

a) Minor of an element :

Consider a square matrix A of order n

Let

$A = \left[a_{ij}\right]_{n \times n}$

The matrix is also can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - & - & - & a_{2n} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & - & - & - & a_{nn} \end{bmatrix}$$

Minor of an element a_{ij} is a determinant of order (nd) by deleting the elements of the matrix A, which are in 6th row and 5th column of A.

E.g. Consider,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

M_{11} = Minor of an element a_{11}

$$A = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

II y

$$M_{12} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{bmatrix}$$

E.g.

(ii) Let,

$$A = \begin{bmatrix} 2 & 5 & 8 \\ 1 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}, M_{12} = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}, M_{13} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}, M_{12} = \begin{bmatrix} 2 & 8 \\ 0 & 6 \end{bmatrix}, M_{23} = \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$$

(b) Cofactor of an element :-

If $A = \left[a_{ij}\right]$ is a square matrix of order n and a_{ij} denotes cofactor of the element a_{ij}.

$C_{ij} = (-1)^{i+j} . M_{ij}$ Where M_{ij} is minor of a_{ij}.

$$\text{If } A = \begin{bmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}$$

$$A_1 = \text{The cofactor of } A_1 = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$B_1 = \text{The cofactor of } b_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$C_1 = \text{The cofactor of } b_1 = (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

E.g. Consider,

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix}$$

$c_{11} = (-1)^{1+1} M_{11}$ $\quad c_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 3 & 6 \end{vmatrix}$

$= (-1)^{1+1} . \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix}$ $\quad = (-1)^3 \times (0-3)$

$= (-1)^{1+1} . \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix}$ $\quad = (-1)^3 \times (0-3)$

$= (1) \times (12-7)$ $\quad = (-1) \times (-3)$

$= (1) \times (12-7)$ $\quad = (-1) \times (-3)$

$= 5$ $\quad = 3$

(C) Cofactor Matrix :-

A matrix $C = [C_{ij}]$ where C_{ij} denotes cofactor of the element a_{ij}. Of a matrix A of order nxn, is called a cofactor matrix.

In above matrix A, cofactor matrix is

$$C = \begin{bmatrix} 5 & 3 & -6 \\ 10 & -6 & 9 \\ -3 & -1 & 2 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{bmatrix}$$

Similarly for a matrix, $A = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$ the cofactor matrix is $c = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

(d) Adjoint of Matrix :-

If A is any square matrix then transpose of its cofactor matrix is called Adjoint of A.

Thus in the notations used,

Adjoint of $A = C^T$

$$\Rightarrow Adj\ A = \begin{bmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{bmatrix}$$

Adjoint of a matrix A is denoted as Adj.A

Thus if,

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix} \text{ than Adj. } A = \begin{bmatrix} 5 & -10 & 3 \\ 3 & -6 & -1 \\ -6 & 9 & 2 \end{bmatrix}$$

Note :

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ than Adj . $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(d) Inverse of a square Matrix:-

Two non-singular square matrices of order n A and B are said to be inverse of each other if,

AB=BA=I, where I is an identity matrix of order n.

Inverse of A is denoted as A^{-1} and read as A inverse.

Thus

$AA^{-1}=A^{-1}A=I$

Inverse of a matrix can also be calculated by the Formula.

$A^{-1} = \frac{1}{|A|}$ Adj.A where $|A|$ denotes determinant of A.

Note:- From this relation it is clear that A^{-1} exist if and only if $|A| \neq 0$ i.e A is non singular matrix.

1.3 ILLUSTRATIVE EXAMPLES

Example 1: Find the inverse of the matrix by finding its adjoint

$$A=\begin{bmatrix}2 & 1 & 3\\ 3 & 1 & 2\\ 1 & 2 & 3\end{bmatrix}$$

Solution: We have,

$$|A|=2\ (3-4)-1\ (9-2)+3\ (6-1)$$
$$=-2-7+15$$
$$|A|=6$$
$$|A|\neq 0$$
A^{-1} exists

Transpose of matrix A=A1

$$\therefore\quad A^{1}=\begin{bmatrix}2 & 3 & 1\\ 1 & 1 & 2\\ 3 & 2 & 3\end{bmatrix}$$

We find co-factors of the elements of A1 (Row-wise)

$$\begin{aligned}&C.F.(2)=-1,\quad C.F.(3)=\ 3,\quad C.F.(1)=-1\\ \therefore\ &C.F.(1)=-7,\quad C.F.(1)=\ 3,\quad C.F.(2)=-5\\ &C.F.(3)=5,\quad C.F.(2)=-3,\quad C.F.(3)=-1\end{aligned}$$

$$\therefore\quad \text{adj}\ (A)=\begin{bmatrix}-1 & 3 & -1\\ -7 & 3 & -5\\ 5 & -3 & -1\end{bmatrix}$$

$$\therefore\quad A^{-1}=\frac{1}{|A|}\ \text{adj (A)}\ =\ \frac{1}{6}\begin{bmatrix}-1 & 3 & -1\\ -7 & 3 & -5\\ 5 & -3 & -1\end{bmatrix}$$

Example 2: Find the inverse of matrix A by Adjoint method, if

$$A=\begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{bmatrix}$$

Solution: Consider

$$|A| = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= 0(-1) - 1(-8) + 2(-5)$$
$$= 0 + 8 - 10$$
$$= -2$$

Co factor of the elements of A are as follows

$$C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$$

$$C_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8$$

$$C_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

$$C_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6$$

$$C_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3$$

$$C_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

$$C_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2$$

$$C_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

Thus,

Cofactor of matrix $C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & 1 \end{bmatrix}$

And Adjoint of A= C^1

$$= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix} \Rightarrow A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix}$$

Note:- A Rectangular matrix does not process inverse.

Properties of Inverse of Matrix:-

i) The inverse of a matrix is unique i.e

ii) The inverse of the transpose of a matrix is the transpose of inverse i.e. $(A^T)^{-1} = (A^{-1})^T$

iii) If A & B are two non-singular matrices of the same order $(AB)^{-1} = B^{-1}A^{-1}$

This property is called reversal law.

Definition:-Orthogonal matrix.:-

If a square matrix it satisfies the relation $AA^T = I$ then the matrix A is called an orthogonal matrix. &

$A^T = A^{-1}$

Example 3:

show that $A = \begin{bmatrix} Cos\theta & Cos\theta \\ Sin\theta & Cos\theta \end{bmatrix}$ is orthogonal matrix.

Solution:

To show that A is orthogonal i.e To show that $AA^T = I$

$$A = \begin{bmatrix} Cos\theta & Sin\theta \\ -Sin\theta & Cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} Cos\theta & Sin\theta \\ -Sin\theta & Cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} Cos\theta & Sin\theta \\ -Sin\theta & Cos\theta \end{bmatrix}\begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} Cos^2\theta + Sin^2\theta & -Cos\theta Sin\theta + Sin\theta Cos\theta \\ -Sin\theta Cos\theta + Cos\theta Sin\theta & Sin^2\theta + Cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is an orthogonal matrix.

Check Your Progress:

Q. 1) Find the inverse of the following matrices using Adjoint method, if they exist.

i) $\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix}$, ii) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$, iii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$,

iv) $\begin{vmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{vmatrix}$, v) $\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$, vi) $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{vmatrix}$

vii) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix}$

Q.3) If A = $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$, B = $\begin{vmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{vmatrix}$, C= $\begin{vmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{vmatrix}$,,

prove that A= B.C^{-1}

Q. 4) If A = $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, prove that Adj. A= A

Q. 5) If A = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, verify if $(Adj.A)^{1} = (Adj.A^{1})$

Q.6) Find the inverse of A = $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 2 & 3 \end{bmatrix}$, hence find inverse of

A = $\begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 6 & 6 & 9 \end{bmatrix}$

1.4 RANK OF A MATRIX

a) Minor of a matrix

Let A be any given matrix of order mxn. The determinant of any submatrix of a square order is called minor of the matrix A.

We observe that, if „r‟ denotes the order of a minor of a matrix of order mxn then $1 \le r \le m$ if m<n and $1 \le r \le n$ if n<m.

e.g. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 4 & 0 & 1 & 7 \\ 8 & 5 & 4 & -3 \end{bmatrix}$$

The determinants

$$\begin{bmatrix} 1 & 3 & -1 \\ 4 & 0 & 1 \\ 8 & 5 & 4 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & 7 \\ 5 & 4 & -3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 4 \\ 4 & 1 & 7 \\ 8 & 4 & -3 \end{bmatrix},$$

$$\begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 5 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 4 \\ 0 & 7 \end{vmatrix}, |1|, |0|, |-3|,$$

Are some examples of minors of A.

b) Definition – Rank of a matrix:

A number „r‟ is called rank of a matrix of order mxn if there is almost one minor of the matrix which is of order r whose value is non-zero and all the minors of order greater than „r‟ will be zero.

e.g.(i) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

Consider e.g. Let

$$A_1 = \begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4,\ A_2 = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8 \text{ etc.}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix} = 1(23) + 2(-2) = 19 \neq 0$$

$\therefore$ Rank of A= 3

(ii) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Here,

$$A_1 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{vmatrix} = 1(1) - 1(-1) + 2(-1) = 0$$

$$A_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$$

Thus minor of order 3 is zero and atleast one minor of order 2 is non-zero
$\therefore$ Rank of A = 2.

Some results:

(i) Rank of null matrix is always zero.

(ii) Rank of any non-zero matrix is always greater than or equal to 1.

(iii) If A is any mxn non-zero matrix then Rank of A is always equal to rank of A.

(iv)Rank of transpose of matrix A is always equal to rank of A.

(v)Rank of product of two matrices cannot exceed the rank of both of the matrices.

(vi)Rank of a matrix remains unleasted by **elementary transformations**.

Elementary Transformations:

Following changes made in the elements of any matrix are called elementary transactions.

(i)Interchanging any two rows (or columns) .

(ii)Multiplying all the elements of any row (or column) by a non-zero real number.

(iii) Adding non-zero scalar multitudes of all the elements of any row (or columns) into the corresponding elements of any another row (or column).

Definition:- Equivalent Matrix:

Two matrices A & B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order & the same rank. It can be denoted by

[it can be read as A equivalent to B] Example

4: Determine the rank of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Solution:

$$\text{Given } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1 \quad \& \quad R_3 \Rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Here two column are Identical . hence 3rd order minor of A vanished

Hence 2nd order minor $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$

$\therefore e(A) = 2$

Hence the rank of the given matrix is 2.

1.5 CANONICAL FORM OR NORMAL FORM

If a matrix A of order mxn is reduced to the form $\begin{bmatrix} I_r & o \\ o & o \end{bmatrix}$ using a sequence of elementary transformations then it called canonical or normal form. Ir denotes identity matrix of order „r" .

Note:-

If any given matrix of order mxn can be reduced to the canonical form which includes an identity matrix of order „r" then the matrix is of rank „r".

e.g. (1) Consider

Example 5: Determine rank of the matrix. A if

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & 6 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_2 - 7R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 33 & 66 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_1 - R_2,\ R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 28 & -56 \end{bmatrix}$$

$R_3 \times \frac{1}{28}$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_1 + 32R_3,\ \ R_2 - 33R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\sim [I_3 \quad o]$

$\therefore$ Rank of A=3

Example 6: Determine the rank of matrix

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 3R_2$

$$\sim\begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

$C_2 - 2C_1$

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

$C_1 \leftrightarrow C_3$

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

$\sim\begin{bmatrix}I_2 & 0\end{bmatrix}$

$\therefore$ Rank of A= 2

Example 7: Determine the rank of matrix A if

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\2 & 3 & -1 & -1\\3 & 1 & 3 & -2\\6 & 3 & 0 & -7\end{bmatrix}$$

Solution:

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\2 & 3 & -1 & -1\\3 & 1 & 3 & -2\\6 & 3 & 0 & -7\end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1,\ R_4 - 6R_1,$

$$\sim\begin{bmatrix}1 & -1 & -2 & -4\\0 & 5 & 3 & 7\\0 & 4 & 9 & 10\\0 & 9 & 12 & 17\end{bmatrix}$$

$R_2 - R_3$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_1 + R_2,\ R_3 - 4R_2,\ R_4 - 9R_2$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \times \frac{1}{11}$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 - C_4$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -7 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 + R_3,\ R_2 + 3R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 - (5C_1 + 3C_2 + 2C_2)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of A= 3

Check Your Progress:-

Reduce the following to normal form and hence find the ranks of the matrices.

i) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ ii) $\begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$ iii) $\begin{vmatrix} -3 & 4 & 6 \\ 5 & -5 & 7 \\ 3 & 1 & -4 \end{vmatrix}$

iv) $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{vmatrix}$ v) $\begin{vmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$ vi) $\begin{vmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{vmatrix}$

vii) $\begin{vmatrix} 2 & 6 & -2 & 6 & 10 \\ -3 & 3 & -3 & -3 & -3 \\ 1 & -2 & 4 & 3 & 5 \\ 2 & 0 & 4 & 6 & 10 \\ 1 & 0 & 2 & 3 & 5 \end{vmatrix}$ viii) $\begin{vmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{vmatrix}$

1.6 NORMAL FORM PAQ

If A is any mxn matrix „r‟ then there exist non singular matrices P and Q such that,

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

We observe that, the matrix A can be expressed as

A = Im In …………(i)

Where Im In are the identity matrices of order m and n respectively. Applying the elementary transformations on this equation. A in L.H.S. can be reduced to normal form. The equation can be transformal into the equations.

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ \text{…………(ii)}$$

Note that, the row operations can be performed simultaneously on L.H.S. and prefactor (i.e. Im in equation (i)) and column operations can be performed simultaneously on L.H.S. and post factor in R.H.S. i.e. [(In in eqn (i)]

Examples 8: Find the non-singular matrices P and Q such that PAQ is in normal and hence find the rank of A.

i) $$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix}$$

Solution: Consider

$A = I_3 A I_3$

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 5 & -4 \\ 3 & 4 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 5C\ ,\ C_3 + 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -11 & -11 \\ 2 & -11 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -11 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_1,\ R_3 - 2R_1,$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -11 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 + C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -11 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \times \frac{1}{11}$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1/11 & 0 & -2/11 \end{bmatrix} A \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1/11 & 0 & 2/11 \\ -1 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1/11 & 0 & 2/11 \\ -1 & 1 & -1 \end{bmatrix} \quad \Delta \quad |P| = \frac{-1}{11}$$

$$Q = \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \Delta \quad |Q| = 1$$

P and Q are non-singular matrices
Also Rank of A = 2

ii) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

Solutions:

Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - C_1, \; C_3 - C_1, \; C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1, \; R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 6R_3,$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 56 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \times \frac{1}{28}, \; R_3 \times (-1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 3/14 & 1/28 & 9/28 \\ -1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix},\ |P| = \frac{1}{28}$$

$$Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ |Q| = 1$$

$\therefore$ P&Q are non singular.

Also,

Rank of A = 3.

Check Your Progress:

A) Find the non-singular matrices P and Q such that PAQ is in normal form and hence find rank of matrix A.

i) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

iv) $\begin{bmatrix} 2 & 3 & 4 & 7 \\ -3 & 4 & 7 & -9 \\ 5 & 4 & 6 & -5 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \\ 15 & 27 & 39 & 51 \\ 6 & 12 & 18 & 24 \end{bmatrix}$

1.7 LET US SUM UP

- Definition of matrix & its types.
- Using Adjoint method to find the A^{-1} by using formula $A^{-1} = \frac{1}{|A|} adjA$
- Rank of the matrix using row & column transformation
- Using canonical & normal form to find Rank of matrix.

1.8 UNIT END EXERCISE

1) Find the inverse of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ if exists.

ii) Find Adjoint of Matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$

iii) Find the inverse of A by adjoint method if $A=\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 2 & 3 & 1 & 0 \end{bmatrix}$

iv) Find Rank of matrix $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

v) Prove that the matrix $A=\begin{bmatrix} Cos\_ & -Sin\_ & 0 \\ Sin\_ & Cos\_ & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is orthogonal

Also find A^{-1}.

vi) Reduce the matrix $A=\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ to the normal form $\_$ and find its rank.

vii) Find the non singular matrix ρ *and* α. such that $\rho A \alpha$ is the normal form when $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Also find the rank of matrix B

$X=\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ & $Y=\begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

viii) Under what condition the rank of the matrix will be 3!

$A=\begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & a \end{bmatrix}$

ix) If $X = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ & $Y = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

Then show that $\circ(xy) \neq \circ(yx)$ where $\circ$ denotes Rank.

x) Find the rank of matrix $A = \begin{bmatrix} 8 & 3 & 6 & 1 \\ -1 & 6 & 4 & 2 \\ 7 & 9 & 10 & 3 \\ 15 & 12 & 16 & 4 \end{bmatrix}$

*****

2

LINEAR ALGEBRIC EQUATIONS

UNIT STRUCTURE

1. Objectives
2. Introduction
3. Canonical or echelon form of matrix
4. Linear Algebraic Equations
5. Let Us Sum Up
6. Unit End Exercise

2.1 OBJECTIVES

After going through this chapter you will be able to

- Find the rank of Matrix.
- Find solution for linear equations.
- Type of linear equations.
- Find solution for Homogeneous equations.
- Find solution of non-Homogeneous equations.

2.2 INTRODUCTION

In XIIth we have solved linear equations by using method of reduction also by rule. Here we are going to find solution of homogeneous and non-homogeneous both with different case. Using matrix we can discuss consistency of system of equation.

2.3 CACONICAL OR ECHOLON FORM OF MATRIX

Let A be a given matrix. Then the canonical or Echelon form of A is a matrix in which

(i)One or more elements in each of first r-rows are non-zero and these first r-rows form an upper triangular matrix.

(ii) The elements in the remaining rows are zero.

Note :

1) The number of non-zero rows in Echelon form is the rank of the matrix.

2) To reduce the matrix to Echelon form only row transformations are to be applied.

Solved Examples :-

Example 1: Reduce the matrix to Echelon and find its rank.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ -1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ -1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2\,R_1$

$R_3 \Rightarrow R_3 - 3\,R_1$

$R_4 \Rightarrow R_4 - 6\,R_1$

$$A=\begin{bmatrix}1 & -1 & 2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 9 & 12 & 17\end{bmatrix}$$

$R_3 \Rightarrow R_3 - \frac{4}{5}\ R_2$

$R_4 \Rightarrow R_4 - \frac{9}{5}\ R_2$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 0 & 33/5 & 22/5\\ 0 & 0 & 33/5 & 22/5\end{bmatrix}$$

$R_4 \Rightarrow R_4 - \ R_3$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 0 & 33/5 & 22/5\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\therefore$ Rank of A = e(A)

= *No. of* non-zero rows

= 3

Check Your Progress:

1) Find the rank of the following matrices by reducing to Echelon form.

i) $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 7\\ 3 & 6 & 10\end{bmatrix}$ *Ans* : 2

ii) $A=\begin{bmatrix}1 & 2 & -1 & 3\\ 3 & 4 & 0 & -1\\ -1 & 0 & -2 & 7\\ 1 & 2 & 3 & -1\end{bmatrix}$

iii) $A=\begin{bmatrix}3 & 4 & 1 & 1\\ 2 & 4 & 3 & 6\\ -1 & -2 & 6 & 4\\ 1 & -1 & 2 & -3\end{bmatrix}$ Ans : 4

2.4 LINEAR ALGEBRIC EQUATIONS

i) Consider a set of equations :

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The equation can be written in the matrix form as :

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A X D

i.e. AX = D

Now we join matrices A and D

$$[A : D] = \begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$$

It is called as Augment matrix

We reduce (A.D.) to Echelon form and thereby find the ranks of A and (A:D)

1) If $^\circ(A) \neq {}^\circ(AD)$ then the system is inconsistent i.e. it has no solution.

2) If $^\circ(AD) = {}^\circ(A)$ then the system is consistent and if

(i) $^\circ(AD) = {}^\circ(A) =$ Number of unknowns then the system is consistent and has unique solution.

(ii) $^\circ(AD) = {}^\circ(A) <$ Number of unknowns and has infinitely many solutions.

Non- Homogeneous equation:-

System of simultaneous equation in the matrix form is AX=D…..(I)

Pre-multiplying both sides of I by A^{-1} we set

$\therefore A^{-1}AX = A^{-1}D$
$\therefore IX = A^{-1}B$
$\therefore X = A^{-1}B$
which is required solution of the given non-homogeneous equation.

Homogeneous linear equation:-

Consider the system of simultaneous equations in the matrix form.
$AX = D$
If all elements of D are zero

i.e

then the system of equation is known as homogeneous system of equations.

In this case coefficient matrix A and the augmented matrix [A,O] are the same. So The rank is same. It follow that the system has solution

$x_1, x_2, x_3 ....x_4 = 0,$ which is called a trivial solution.

Example 2: Solve the following system of equations

$$2x_1 - 3x_2 + x_3 = 0$$
$$x_1 + 2x_2 - 3x_3 = 0$$
$$4x_1 - x_2 - 2x_3 = 0$$

Solution: The system is written as

$$AX = 0$$
$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence the coefficient and augmented matrix are the same We consider

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}$$

$R_1 \Rightarrow R_1 \leftrightarrow R_2$

$$A=\begin{bmatrix}1 & 2 & -3\\ 2 & -3 & 1\\ 4 & -1 & -2\end{bmatrix}$$

$R_2 \Rightarrow R_2$-2 R_1 & $R_3 \Rightarrow R_3-4R_1$

$$=\begin{bmatrix}1 & 2 & -3\\ 0 & -7 & 7\\ 0 & -9 & -10\end{bmatrix}$$

$R_2 \Rightarrow R_2\times {}^{1}\!/_{7}$

$$=\begin{bmatrix}1 & 2 & -3\\ 0 & 1 & -1\\ 0 & -9 & -10\end{bmatrix}$$

$R_3 \Rightarrow R_3$+9R_2 & $R_1 \Rightarrow R_1-2R_2$

$$=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & -19\end{bmatrix}$$

$R_3 \Rightarrow R_3\times {}^{-1}\!/_{7}$

$$=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$$

$R_2 \Rightarrow R_2$+R_3 & $R_1 \Rightarrow R_1+R_3$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Hence Rank of A is 3

$\therefore \ell(A)=3,$

The coefficient matrix is non-singular

Therefore there exist a trivial solution

$x_1 = x_2 = x_3 = 0$

Example 3: Solve the following system of equations

$x_1 + 3x_2 - 2x_3 = 0$

$2x_1 - x_2 + 4x_3 = 0$

$x_1 - 11x_2 + 14x_3 = 0$

Solution: The given equations can be written as

$AX = 0$

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & 11 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the coefficient & augmented matrix are the same

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$R_2 \Rightarrow R_2$-2 R_1 & $R_3 \Rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Here rank of A is 2 i.e

$$\ell(A) = 2$$

So the system has infinite non-trivial solutions.

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_2 - 2x_3 = 0$

$-7x_2 - 8x_3 = 0$

$7x_2 = 8x_3$

$x_2 = \frac{8}{7} x_3$

Let $x_3 - 8x_3 = a$

$\therefore x_2 = \frac{8}{7} a$

$\therefore x_1 + 3\left(\frac{8}{7} a\right) - 2a = 0$

$\therefore x_1 + \frac{24}{7} a - 2a = 0$

$\therefore x_1 = 2a - \frac{24}{7} a$

$\therefore x_1 = -\frac{10}{7} a$

Hence $x_1 = -\frac{10}{7}a$ $x_2 = \frac{8}{7}a$ and $x_3 = a$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{10}{7}a \\ \frac{8}{7}a \\ a \end{bmatrix}$$

Hence infinite solution as deferred upon value of a

Example 4: Discuss the consistency of

$$2x + 3y - 4z = -2$$
$$x - y + 3z = 4$$
$$3x + 2y - z = -5$$

Solution: In the matrix form

$$\begin{bmatrix} 2 & 3 & -4 \\ 1 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -5 \end{bmatrix}$$

Consider an Agumental matrix

$$[A:D] = \begin{bmatrix} 2 & 3 & -4 & : & -2 \\ 1 & -1 & 3 & : & 4 \\ 3 & 2 & -1 & : & -5 \end{bmatrix}$$

$$R_2 \to R_2 - \frac{1}{2} R_1$$

$$R_3 \to R_3 - \frac{3}{2} R_1$$

$$[A:D] = \begin{bmatrix} 2 & 3 & -4 & : & -2 \\ 0 & -5/2 & 5 & : & 5 \\ 0 & -5/2 & 5 & : & -2 \end{bmatrix}$$

$$R_2 \to R_3 - R_2$$

$$[A:D] = \begin{bmatrix} 2 & 3 & -4 & : & -2 \\ 0 & -5/2 & 5 & : & 5 \\ 0 & 0 & 5 & : & -7 \end{bmatrix}$$

$$\therefore\ ^{\circ}(AD)=3$$

$$^{\circ}(A)=2$$

$$\therefore\ ^{\circ}(AD)\neq\ ^{\circ}(A)$$

$\therefore$ The system is inconsistent and it has no solution.

Example 5: Discuss the consistency of

$$3x+y+2z=3$$
$$2x-3y-z=-3$$
$$x+2y+z=4$$

Solution: In the matrix form,

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$A \qquad X = D$$

Now we join matrices A and D

Consider

$$[A:D] = \begin{bmatrix} 3 & 1 & 2 & : & 3 \\ 2 & -3 & -1 & : & -3 \\ 1 & 2 & 1 & : & 4 \end{bmatrix}$$

We reduce to Echelon form

$$R_1 \to R_3$$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -3 & -1 & : & -3 \\ 3 & 1 & 2 & : & 3 \end{bmatrix}$$

$$R_2 \to R_2 - 2\,R_1$$
$$R_3 \to R_3 - 3\,R_1$$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -7 & -3 & : & -11 \\ 0 & -5 & -1 & : & -9 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{5}{7}\,R_2$$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & 0 & 8/7 & : & -8/7 \end{bmatrix} ....(1)$$

This is in Echelon form

$$\therefore \,{}^\circ (AD) = 3$$

$$^\circ (A) = 3$$

$\therefore \; ^\circ \; (AD) = {}^\circ \; (A) =$ Number of unknowns

$\therefore$ system is consist and has unique solution.

Step (2) : To find the solution we proceed as follows. At the end of the row transformation the value of z is calculated then values of y and the value of x in the last.

The matrix in e.g. (1) in Echelon form can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 8/7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -8/7 \end{bmatrix}$$

$\therefore$ Expanding by R_3

$$\frac{8}{7} Z = -8/7$$

$\therefore$ z = -1

$\therefore$ expanding by R_2

$$-7y - 3z = -11$$

$$-7y - 3(-1) = -11$$

$$-7y + 3 = -11$$

$$\cancel{-}7y = \cancel{-}14$$

$$y = 2$$

exp*anding* by R_1

$$x + 2y + z = 4$$

$$x + \cancel{4} - 1 = \cancel{4}$$

$\therefore$ x = 1

$\therefore$ x = 1, y = 2, z = -1

Example 6: Examine for consistency and solve

$$5x + 3y + 7z = 4$$

$$3x + 26y + 2z = 9$$

$7x+2y+10z=5$

Solution:

Step (1) : In the matrix form

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$A \qquad X = D$$

Consider

$$[A:D]=\begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$R_1 \rightarrow \frac{1}{5} R_1$$

$$[A:D]=\begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 - 7R_1$

$$[A:D]=\begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 0 & 121/5 & -11/5 & : & 33/5 \\ 0 & -11/5 & 1/5 & : & -3/5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + \frac{1}{11} R_2$$

$$[A:D]=\begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 0 & 121/5 & -11/5 & : & 33/5 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\therefore \quad ^\circ(AD)=2$

$\quad ^\circ(A)=2$

$\therefore \quad ^\circ(AD)= {}^\circ(A)=2<3=$ *Number* of unknowns

The system is consistent and has infinitely many solutions.

Step (2) :- To find the solution we proceed as follows:

Let

$z = k.......[k = \text{ parameter}]$

$\therefore$ By expanding R_2

$121/5y - 11/5z = 33/5$

$\therefore$ 11y-z $= 3$

$\therefore$ $y = \frac{z+3}{11}$

$\therefore$ put $z = k$

$\therefore$ $y = \frac{k+3}{11}$

By exapanding R_1

$x + 3/5\, y + 7/5\, z = 4/5$

$\therefore$ $x = \frac{7}{11} - \frac{16k}{11}$

Check Your Progress:

Solve the system of equations:

i) $2x_1 + x_2 + 2x_3 + x_4 = 6$

$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$

$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$

$2x_1 + 2x_2 - x_3 + x_4 = 10$

Ans : consistent

ii) $x_1 = 2, \ x_2 = 1, \ x_3 = -1, \ x_4 = 3$

$2x_1 + x_2 + x_3 + x_4 = 2$

$3x_1 - x_2 + x_3 - x_4 = 2$

$x_1 + 2x_2 - x_3 + x_4 = 1$

$6x_1 + 2x_2 + x_3 + x_4 = 5$

Ans : Infinitely many solutions,

iii) $x_1 = k, \ x_2 = 3 - 4k, \ x_3 = 2 - \frac{5}{2}k, \ x_4 = \frac{9}{2}k - 3$

3 $x_1 + x_2 + x_3 = 4$

$2x_1 + 5x_2 - 2x_3 = 3$

$x_1 + 7x_2 - 7x_3 = 5$

Ans : Inconsistent

iv) $x_1 - x_2 - x_3 = 0$

$x_1 + 2x_2 - x_3 = 0$

$2x_1 + x_2 + 3x_3 = 0$

Ans: Trivial Solution.

v) $x_1 + 2x_2 + 3x_3 = 0$

$2x_1 + 4x_2 + 7x_3 = 0$

$3x_1 + 6x_2 + 10x_3 = 0$

Ans : Definitely many solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ \frac{-1}{2} a \\ 0 \end{bmatrix}$$

2.5 LET US SUM UP

In this chapter we have learn

- Using row echelon from finding Rank of matrix.
- Representing linear equation m x n in to argumented matrix.
- Consistency of matrix.
- Solution of Homogeneous equations.
- Solution of non homogeneous equations.

2.6 UNIT END EXERCISE

1) Reduce the following matrix in Echolon form & find its Rank.

i) $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$ *Ans : Rank = 2*

ii) $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ *Ans : Rank = 3*

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ *Ans : Rank = 2*

iv) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ *Ans : Rank = 2*

2) Solve the following system of equations.

i) $x_1 + x_2 + x_3 = 3,\ x + 2x_2 + 3x_3 = 4,\ x_1 + 4x_2 + 9x_3 = 6$

Ans:- $x = 2, y = 1, z = 0.$

ii) $2x_1 - x_2 - x_3 = 0,\ x_1 - x_3 = 0,\ 2x_1 + x_2 - 3x_3 = 0$

Ans:- $x_1 = x_2 = x_3 = a \ldots\ldots \therefore a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

iii) $5x_1 - 3x_2 - 7x_3 + x_4 = 10$

$-x_1 + 2x_2 + 6x_3 - 3x_4 = -3$

$x_1 + x_2 + 4x_3 - 5x_4 = 0$

iii) $2x_1 + 3x_2 - 2x_3 = 0$

$3x_1 - x_2 + 3x_3 = 0$

$7x_1 + 5x_2 - x_3 = 0.$

iv) $x_1 - 4x_2 - x_3 = 3$

$3x_1 + x_2 - 2x_3 = 7$

$2x_1 - 3x_2 + x_3 = 10.$

v) $x_1 - 4x_2 + 7x_3 = 8$

$3x_1 + 8x_2 - 2x_3 = 6$

$7x_1 - 8x_2 + 26x_3 = 31$

*****

3

LINEAR DEPENDANCE AND INDEPENDANCE OF VECTORS

UNIT STRUCTURE

3.1 Objectives

After going through this chapter you will able to

- Find linearly independent & linearly dependent vector.
- Inner product of two vector
- Find characteristic equation of matrix
- Find the of characteristic equation i.e
- Find the corresponding .Eigen vector to Eigen value.

2. Introduction

In this chapter we are going to discuss linearly dependent & independent also. Inner two vector using the characteristic equation of matrix. We are going to evaluate .Eigen value & Eigen.vector of matrix A.

Vector :- An set of n elements written as $x = [x_1, x_2, x_3, x_4, ...x_n]$ is called a vector of n-dimensions.

Note : Any two or column matrix is called as a vector and numbers are called as scalars.

3. Definitions

Linearly Independent Vector
Let
Let $x_1, x_2, ...x_n$ be n vectors of some order

Let $c_1x_1 + c_2x_2 + ... + c_nx_n = 0$

Where $c_1,\ c_2,$ ...... are scalars.

If (i) $c_1 = c_2 = ... = c_n = 0$ then

$x_1,\ x_2,\ ....x_n$ are linearly independent

and (ii) if not all c_i are zero then $x_1, x_2, ... x_n$ are linearly dependent

If $x_1, x_2, ...x_n$ are linearly dependent then a relation exists between them which can be found out

Solved examples:-

Example 1: Examine for linear dependence

$x_1 = (1\ \ 2\ \ 4)^T\ ,\ \ x_2 = (3\ \ 7\ \ 10)^T$

Solution: We have,

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix},\quad x_2 = \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix}$$

$$Let\ \ c_1x_1 + c_2x = 0$$

$$i.e.\quad c_1\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e.\quad \begin{bmatrix} c_1 & + & 3c_1 \\ 2c_1 & + & 7c_2 \\ 4c_1 & + & 10c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\quad c_1 + 3c_2 = 0$$

$$2c_1 + 7c_2 = 0$$

$$4c_1 + 10c_2 = 0$$

Consider first two equations in matrix form.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A\quad X\quad = 0$$

$$\therefore\quad |A| = 7 - 6$$

$$|A| = 1$$

$\therefore\quad |A| \neq 0$

$\therefore$ system has zero solution.

i.e. $c_1 = c_2 = 0$

$\therefore\quad x_1,\ x_2$ are linearly independent

Example 2: Examine for linear dependence.

$x_1 = (1 \quad 2 \quad 3)^T$, $x_2 = (3 \quad -2 \quad 1)^T$, $x_3 = (1 \quad -6 \quad -5)^T$

Solution:

Step (1) We have

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$$

$$Let \quad c_1 x_1 + c_{21} x_2 + c_3 x_3 = 0$$

$$\therefore \quad c_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} c_1 & +3c_2 & +c_3 \\ 2c_1 & -2c_2 & -6c_3 \\ 3c_1 & +c_2 & -5c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad c_1 + 3c_2 + c_3 = 0$$

$$2c_1 - 2c_2 - 6c_3 = 0$$

$$3c_1 + c_2 - 5c_3 = 0$$

Step (ii) In matrix form,

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A X = 0

Consider

$$[A:0] = \begin{bmatrix} 1 & 3 & 1 & : & 0 \\ 2 & -3 & -6 & : & 0 \\ 3 & 1 & -5 & : & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2\,R_1$

$R_3 \rightarrow R_3 - 3\,R_1$

$$[A:0] = \begin{bmatrix} 1 & 3 & 1 & : & 0 \\ 0 & -8 & -8 & : & 0 \\ 0 & -8 & -8 & : & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$R_2 \rightarrow -\frac{1}{8} R_2$

$$[A:0] = \begin{bmatrix} 1 & 3 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$e\ (A\,0) = 2$

$e\ (\mathrm{A}) = 2$

$\therefore\ \ \mathrm{e}\ (\mathrm{A{:}0}) = e\ (\mathrm{A}) = 2 <$ Number of unknowns

$\therefore$ system has non-zero solution

i.e. $\mathrm{c}_1,\ \mathrm{c}_2,\ \mathrm{c}_3$ are non zero

$\therefore\ \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3$ are linearly dependent

Step (iii):

To find relation between

$x_1,\ \mathrm{x}_{2,}\ \mathrm{x}_3$

Let

$\mathrm{c}_3 = k$

By exp*anding* R_2

$\mathrm{c}_2 + \mathrm{c}_3 = 0$

$\therefore\ \ \mathrm{c}_2 = \text{-}\mathrm{c}_3$

$\mathrm{c}_2 = -k$

By expanding R_1

$\mathrm{c}_1 + 3\mathrm{c}_2 + \mathrm{c}_3 = 0$

$\mathrm{c}_1 - 3\mathrm{k} + \mathrm{k} = 0$

$\mathrm{c}_1 = 2k$

$\therefore\ \mathrm{c}_1 x_1 + \mathrm{c}_2 x_2 + \mathrm{c}_3 x_3 = 0$

$\therefore\ 2\mathrm{kx}_1 - \mathrm{kx}_2 + \mathrm{kx}_3 = 0$

$\therefore\ 2\mathrm{x}_1 - \mathrm{x}_2 + \mathrm{x}_3 = 0$ is a relation.

Check your progress:

1) Show that the vectors $x_1 = (1\ \ 1\ \ \ 1\ ),\ \ \mathrm{x}_2\,(1,\ 2,\ 3), \mathrm{x}_3\,(\ 2,\ 3,\ 8)$ are linearly independent

2) Are the following vectors linearly dependent? If so find the relation

i) $x_1 = (1\ \ 2\ \ \ 4\ ),\ \ \mathrm{x}_2 = (2,\ \text{-}1,\ 3), \mathrm{x}_3 = (\ 0,\ 1,\ 2), \mathrm{x}_4 = (\ -3,\ 7,\ 2)$

Ans : Dependent $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$

(ii) $x_1 = (2\ \ \text{-}1\ \ \ 3\ \ 2),\ x_2 = (1\ \ 3\ \ 4\ \ 2),\ \ x_3 = (3\ \ \text{-}5\ \ \ 2\ \ 2)$

Ans :- Dependent, $2x_1 - x_2 - x_3 = 0$

(iii) $x_1 = (1\ \ 1\ \ \ 1\ \ \ \ 3),\ x_2 = (1\ \ 2\ \ 3\ \ 4),\ \ x_3 = (2\ \ \ 3\ \ 4\ \ 9)$

Ans : Independent.

3.4 THE INNER PRODUCT

If $X=(x_1,x_2.......x_n)$ and $Y=(y_1,y_2.......y_n)$
then<X,Y> denotes inner product
$<X,Y>=x_1y_1+x_2y_2+x_3y_3+........+x_ny_n$ is in inner product of X and Y.
Let V be a vector space and X,Y V then<X,Y> it said to be an inner product if it satisfies following properties.

i) <X,Y> =0
ii) <X,Y> = <Y,X>
iii) <X,Y+Z> = <X,Y> + <X,Z>
iv) <X,~ Y> = ~ <X,Y> where ~ is scalar.
v) <X,Y> = 0 if and only if X=0.

Example 3: Show that $<X,Y>=x_1y_1+2x_2y_2+4x_3y_3$
Satisfies all properties of inner product

Solution: $<X,Y>=x_1y_1+2x_2y_2+4x_3y_3$

i) $<X,Y>=x_1y_1+2x_2y_2+4x_3y_3$

$$=(x_1)^2+2(x_2)^2+4(x_3)^2\geq 0$$

$$<X,Y>\geq 0$$

$$<X,Y>=0(x_1)^2+2(x_2)^2+4(x_3)^2=0$$

$$x_1=0,x_2=0,orx_3=0$$

$$\therefore <X,X>0x=0$$

ii) $<X,Y>=x_1y_1+2x_2y_2+4x_3y_3$

$$=y_1x_1+2y_2x_2+4y_3x_3$$

$$=<Y,X>$$

iii) $<X,Y+Z>=x_1(y_1+z_1)+2x_2(y_2+z_2)+4x_3(y_3+z_3)$

$$=x_1y_1+x_1z_1+2x_2y_2+2x_2z_2+4x_3y_3+4x_3z_3$$

$$=x_1y_1+2x_2y_2+4x_3y_3+x_1z_1+2x_2z_2+4x_3z_3$$

$$=<X,Y>+<X,Z>$$

iv) $<X,\sim Y> = x_1(\sim y_1)+2x_2(\sim y_2)+4x_3(\sim y_3)$

$$=\sim x_1y_1+\sim 2x_2y_2+\sim 4x_3y_3$$

$$=\sim(x_1y_1+2x_2y_2+4x_3y_3)$$

$$=\sim <x,y>$$

Here all properties are satisfied
$\therefore <X,Y>$ is an inner product.

Check Your Progress:

Prove all the properties of an inner product for the following:-

i. $<X,Y>=16x_1y_1-25x_2y_2$
ii. $<X,Y>=8x_1y_1+x_2y_2-x_3y_3$

iii. $< X, Y > = 3x_1y_1 - x_2y_2 - 4x_3y_3$

iv. $< f, g > = \int_a^b f(t).g(t).dt$

3.5 Eigen Values And Eigen Vectors

Definition:-

Let A be a given square matrix.
Then there exists a scalar λ and non-zero vector X such that

$$AX = \lambda X \text{..........(1)}$$

Our aim is to find and x for given matrix A using equation (1)

λ is called as eigen value, latent roots of a matrix value, characteristic value or root of a matrix A and x is called as eigen vector or characteristic vector etc.

X is a column matrix

Method of finding λ and x :-

We have,

$$AX = \lambda X$$

$$\therefore \quad AX - \lambda IX = 0 \text{.....} \quad [x = IX, I = \text{unit matrix}]$$

$$\therefore \quad (A - \lambda I)X = 0 \text{.............(2)}$$

Equation 2 is a set of homogenous equation and for non-zero x, we have

$$|A - \lambda I| = 0 \text{..........(3)}$$

This equation is called the characteristic equation of

First we solve equation (3) to find eigen values or roots. Then we solve equation (2) to find Eigen vectors.

Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

equation (2) i.e. $(A - \lambda I)x = 0$ becomes

$$\left\{ \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i.e.\begin{bmatrix} a_1 - a & b_1 & c_1 \\ a_2 & b_2 - a & c_2 \\ a_3 & b_3 & c_3 - a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (2)$$

and equation (3) i.e. $|A - a\ x| = 0$ is

$$\begin{bmatrix} a_1 - a & b_1 & c_1 \\ a_2 & b_2 - a & c_2 \\ a_3 & b_3 & c_3 - a \end{bmatrix} = 0 \rightarrow (3)$$

Note :

1) Equation (2) is called as matrix equation of A in a

2) Equation (3) is called as characteristic equation of A in a

3) Usually given matrix A is of order 3X3 . Therefore it will have 3 eigen values and for every eigen value there will be corresponding eigen vector which is a column matrix of order 3X1. There are 3 such column matrices.

4) Eigen vectors are linearly independent.

5) Method of finding eigen values is same for any given matrix A.

Method of finding eigen vectors is slightly different and we study 3 types of such problems.

Type (I) : When all eigen values are distinct and matrix A may be symmetric or non- symmetric.
Type (II) : When eigen values are repeated and A is non-symmetric
Type (III) : When eigen values are repeated and A is symmetric.

Solved examples :-

Type (I) : All roots are non- repeated.

Example 4: Find eigen values and given vectors for

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution: *Step* (1) : Charactristic equation of A in a i s

$$|A - a\ I| = 0$$

$$i.e.\begin{vmatrix} 2-a & -2 & 3 \\ 1 & 1-a & 1 \\ 1 & 3 & -1-a \end{vmatrix} = 0$$

$\therefore\ a^3 - (\textit{sum of}\ \text{diagonal elements of A})\ a^2 +$

$(\textit{sum of}\ \text{minors of diagonal elements of A})\ a - |A| = 0$

$\therefore |A| = 2\ (-1-3) + 2\ (-1-1) + 3\ (3-1)$

$= -8-4+6$

$|A| = -6$

Characteristic equation is given by

$$\therefore \lambda^3 - 2\lambda^2 + (-4-5+4)\lambda - (-6) = 0$$

$\therefore \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

$\sin ce$ sum of coefficient $= 0$

$\therefore (\lambda - 1)$ is a factor.

Synthetic division:

1	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0

$\therefore (\lambda - 1)(\lambda^2 - \lambda - 6) = 0$

$\therefore (\lambda - 1)(\lambda - 3) \cdot (\lambda + 2) = 0$

$\therefore \lambda = 1, -2, 3$

$\therefore$ The roots are non- repeated

Step (ii) :- Now we find eigen vectors

Matrix equations is given by

$$(A - \lambda I)X = 0$$

$$i.e. \begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case $(i):-$ When $\lambda = 1$, matrix eq^n becomes

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving first two rows by Cramer"s rule.

We have,

$$x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$\therefore \frac{x_1}{-2} = \frac{-x_2}{-2} = \frac{x_3}{-2}$$

$$\therefore \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore\ x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) :- When ª$_2 = -2$

Matrix equation is given by

$$\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\ \frac{x_1}{-11} = \frac{x_2}{1} = \frac{x_3}{14}$$

$$\therefore\ \frac{x_1}{-11} = \frac{x_2}{-1} = \frac{x_3}{14}$$

$$\therefore\ x_2 = \begin{bmatrix} -11 \\ -1 \\ 14 \end{bmatrix}$$

Case (iii) : When ª $_3 = +3$ matrix equation is given by

$$\begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\ \frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\therefore\ \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{4} \qquad \therefore\ x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Type (II) :- Repeated eigen values and A is non- symmetric.

Example 5: Find eigen values and eigen vectors for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Solution:

Step (1) :- Characteristic equation of A in ª is

$[A -$ ª $I] = 0$

i.e. ª ^{3}-9 ª $^2 + (6+5+4)$ ª -7 = 0

ª ^{3}-9ª ^{2} +15 ª -7 = 0

since sum of co-efficients = 0

$\therefore$ (ª -1) is a factor

synthetic division

1	1	−9	15	−7
		1	−8	7
	1	−8	7	0

$\therefore\ a^2 - 8a + 7$

$= (a-7)(a-1)$

$\therefore\ a^3 - 9a^2 + 15a - 7 = 0$

$\therefore\ (a-1)(a-1)(a-7) = 0$

$a = 7, 1, 1$

Here two roots are repeated. First we find eigen vectors for non-repeated root.

Step II :- Matrix equation of A in a is

$$(A - a\,I)X = 0$$

$$\begin{bmatrix} 2-a & 1 & 1 \\ 2 & 3-a & 2 \\ 3 & 3 & 4-a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i) :- For $a = 7$

Matrix equation is

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\ \frac{x_1}{6} = \frac{-x_2}{-12} = \frac{x_3}{18}$$

$$\therefore\ \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

$$\therefore\ x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Case (ii) :- Let $a = 1$

Matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By cramers‟s rule we get

$$\frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{0}$$

$$i.e.\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But by definition we want non-zero x_2

Thank You